Approximating a multifactor diffusion on a tree.

Andrey Chirikhin*

September 2004

Abstract

A new method of approximating a multifactor Brownian diffusion on a tree is presented. The method is based on local coupling of the marginal branches with the multifactor Gaussian copula. The structure of the copula's correlation matrix is selected so as to correspond to the available correlation parameters in the model. The method is more general than the ones existing in the literature, while it reproduces the limiting cases and it is immediately generalizable to non-standard tree branching if it occurs.

Contents

*andre@chirikhin.com

1 General setup

Let $\mathbf{X} \in \mathbb{R}^n$ follow a k-dimensional generalized Brownian diffusion, where drift μ and diffusion $\Sigma > 0$ are Markov processes from $\mathcal{L}^1$ and $\mathcal{L}^2$:

$$\begin{aligned} &\mathrm{d}\mathbf{X} = \mu(\mathbf{X}, t)\mathrm{d}t + \Sigma(\mathbf{X}, t)\mathrm{d}\mathbf{W}, \\ &\langle \mathrm{d}W_i, \mathrm{d}W_j \rangle = \rho_{ij}\mathrm{d}t, 1 \leq i < j \leq k. \end{aligned} \tag{1.1}$$

We interpret the coordinates $\mathbf{X}$ as the processes of the underlying securities. When dealing with pricing certain derivative contingent on $\mathbf{X}$, such diffusions are often approximated on trees. In what follows we assume that the approximation tree is l nomial recombining, that is i-th coordinate is l_i-nomial recombining tree, and from each node has $\prod_i l_i$ followers.

Two main complications arise when implementing the tree approximation in higher dimension in the non-independent case.

- **Local approximation of the dependence structure.** Standard approach deals with approximating the discrete multivariate transition kernel. The problem is that the number of correlations defining the Brownian motions not enough to fully characterize the joint distribution function, once information that we have joint Gaussian distribution is dropped, as correlations capture only linear dependence. Additional assumptions or approximations, implicit or explicit, are needed to handle higher joint moments.

  The standard approach, followed in [HA93], [FD01], [Sch99] is to approximate each component of $\mathbf{X}$ independently on a l-nomial tree and then couple the margins on each node by solving for the full discrete joint distribution of n^l components, such that pairwise local correlations are reproduced. As we noticed before, already in bivariate trinomial case, or tri-variate binomial case the corresponding linear system appears underspecified that requires making extra assumptions to reduce dimensionality. In the papers mentioned above, the extra condition is reproduction of the perfectly correlated case and assumption that the joint distribution is well approximated as linear combination of independent and perfectly correlated (or anticorrelated) cases, the weight of the latter being proportional to correlation. While this worked for two-dimensional triaomial case, this will not be enough to complete the system in higher dimensional cases.

- **Nonstandard branching.** For the above mentioned method to work, one requires to know the correlation structure in some extreme cases. These can be easily guessed if branching is standard, but not in the case of nonstandard branching, in which case further approximation is needed.

Both problems can be resolved by recalling the fact that we are discretizing multidimensional Brownian diffusion. By definition of the Brownian motion, random vector $(\mathrm{d}\mathbf{X} - \mu(\mathbf{X}, t)\mathrm{d}t)/\Sigma(\mathbf{X}, t)$ follows n-dimensional, k-factor Gaussian distribution, hence the joint distribution of the vector's

component-wise rank transfromaion $\mathbf{p} = prob(\xi < (\mathrm{d}\mathbf{X} - \mu(\mathbf{X}, t)\mathrm{d}t)/\Sigma(\mathbf{X}, t))$ is n-dimensional, k-factor Gaussian copula.

Note, that to make this conclusion we need to start with multivariate Brownian motion in specification of $\mathbf{X}$. Only considering marginal Brownian motions and specifying correlation of their increments is not enough to fully specify the dependence structure, as other than Gaussian copula can reproduce the increments correlation, e.g. t-copula. However, adding the condition for correlation assumes that it correlation is computable, hence that the joint dynamics of the marginal Brownian motions is already specified somehow, which, by Sklar theorem, is equivalent to specifying their copula. Therefore, the joint distribution of ranks of the marginal Brownian montions' paths will have that copula. For this reason, in what follows we restrict the assumption that the copula of the increments is Gaussian, unless specified otherwise.

Before proceeding, we make few comments on why the standard approach yields the problem being underspecified. As we have just noted, by selecting a copula of the marginal Brownian increments, we do **fully** specify the multivarite distribution of $\mathrm{d}\mathbf{X}$. The problem becomes underspecified when, after discretizing the process, we choose to look only at the second moment and ignore the copula that we had started with. By doing so, we partly loose information about the joint dynamics that was present in the original formulation, at the expense of seemingly better analytical tractability. Our observation above shows that with dimensionality increasing, the second moment alone quickly becomes insufficient, and making the approximation tractable will require further assumptions in higher dimensions. Neither would be needed if we could decouple marginal processes, build approximating trees independently and then couple them back fully utilizing the original denendance specification. Using copula of the increment, if it is available and tractable (and it is both for low-dimentional Brownian motion), does exactly that, and, as we will show, works automatically in the case of nonstandard branching.

Specifically, for a given node of a tree denote X_θ the n-dimensional matrix (a 3-d cube if $n = 3$) of the follower nodes, where θ is a n-dimensional coordinate vector: $\theta_i = 1..l_i, i = 1..n$. The matrix X_θ is the direct product of the n marginal state vectors $x_i, i = 1..n$, which are determined by the procedure that would build a one-dimensional tree for the corresponding component of $\mathbf{X}$. In what follows we will designate the j-th component of i-th vector x_{ji} to mimic matrix notation, where the second coordinate corresponds to the "vector column". Note, however, that x_{ji} is not necessarily a matrix, as number of elements in i-th and k-th columns are l_i and l_k correspondingly. As $\mathbf{X}_i \in \mathbb{R}$, which is ordered, we always can assume that $x_{ji} < x_{j+1,i}$.

Building the marginal trees amounts to computing the marginal local transition vectors $p_{ji} = prob(X_i(t+1) = x_{ji})$ for all nodes of $X_i(t)$, while building joint distribution is determining $prob(X(t+1) = X_\theta)$ for all realizations of θ. Introduce the marginal local cumulative probabilities v_{ji}, with first indices now starting at 0 (not 1 as for θ and all other matrices!) such that

$$v_{ji} = \begin{cases} 0 & \text{if j} = 0 \\ \sum_{s=1}^{j} p_{ji} & \text{if j} > 0 \end{cases} \tag{1.2}$$

This will, effectively, make use of the copula "groundness" on the next step and simplify notation

of the main formula (1.3).

Consider an arbitrary n dimensional copula C (for definition and basic properties see [PMS02]). The state transition probability is therefore a discrete "density" of C, given by the "includsion/exclusion" formula, present in the copula definition [PMS02]. For given follower state θ define componentwise two vectors $a_{is}, s = 1, 2$, as function of the "index" vector θ:

$$a_{is} = \theta_i - 2 + s,$$

thus substituting θ_i by $\theta_i - 1$ when $s = 1$.

For a given follower state with index θ, consider the n-dimensional hull, with i-th edge $[v_{i,a_{i,1}}, v_{i,a_{i,2}}]$. Then, the transition probability is given by:

$$prob(X_{t+1} = X_\theta) = \sum_{i_1=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{\sum_{k=1}^{n} i_k} C(v_{1,a_{1,i_1}}, ..., v_{n,a_{n,i_n}}). \tag{1.3}$$

This formula allows recovering the joint transition probabilities for arbitrary marginal distributions and arbitrary probability thresholds. Intuitively, we have first decoupled the margins, which enabled us to discretize the states and determine p_{ji}. Then we build cumulative transition probabilities and couple them back with some copula of the proper dimension, which allows us to recover the transition "density", ie transition probabilities to states. As we did not make any assumptions about spacing of the marginal states, the procedure works for both standard and nonstandard branching.

In the case that we have started with, the copula is n-dimensional (k-factor) Gaussian copula. Note, the independence copula trivially reproduces the independent case, and Gaussian copula also reproduces the extreme case of perfect correlation (by construction of the copula), therefore it reduces to the same extreme cases as considered in [FD01], [Sch99].

As we we did not assume any specific form of the SDE coefficients, the procedure can obviously be generalized for the case of arbitrary marginal Markov diffusions, once the copula of it's increment is known.

The general procedure would therefore be as follows:

1. Build marginal trees using some standard procedure.
2. Order states and compute cumulative transition probabilities following (1.2).
3. For each state, create local transition measures using local copula of the increment. Given marginal thresholds v_{ji}, use formula (1.3) where indices obey shifting (**??**)
4. Perform further linear transformations of the states if needed (as those in the HW procedure that reproduces marginal forward curves).

2 The two-dimensional case

Consider a special case [FD01] when $n = 2, k = 2$, $\mu_i(\mathbf{X}, t) = \mu_i(\mathbf{X}_i, t)$, $\Sigma_{ii}(\mathbf{X}, t) = \sigma_i(\mathbf{X}_i, t)$, and $\Sigma_{i,j\neq i} = 0$. Effectively, we have two "factorized" one-dimensional Browninan motions, increments of which are correlated with constant correlation.

Bivariate Gaussian copula will be given in this case by

$$G(x, y) = \Phi_{\rho \mathrm{d}t}\left(\Phi^{-1}(x), \Phi^{-1}(y)\right),$$

where $\Phi_\rho(,)$ is bivariate standardized Gaussian cumulative distribution, with correlation ρ, and $\Phi^{-1}()$ is one-dimensional standardized inverse Gaussian distribution.

Suppose we enumerate the the marginal follower states of x and y with indices s and r, both taking values $\{1, 2, 3\}$. Then formula (1.3) simplifies into

$$\begin{aligned} prob(X_{t+1} = x_{s,r}) =& G(v_{x,s}, v_{y,r}) + G(v_{x,s-1}, v_{y,r-1}) - \\ & G(v_{x,s}, v_{y,r-1}) - G(v_{x,s-1}, v_{y,r}), \end{aligned} \tag{2.1}$$

where v_{ji} are given by (1.2), and marginal transition probabilities are obtained when building marginal trees.

Note that it is possible to specify local correlation structure, if it is needed.

3 Literature review

From the most abstract point of view, we are dealing with approximation of a Markov process' SDE on a lattice. In addition to vast literature on such methods in the mathematical finance literature, there are have been quite few results on generalizing the setup to the process' increment other than Brownian or some simple form of jump diffusion. Few works consider a general setup of a Levy process, arguing that every martingale diffusion can be represented as Brownian motion with changed time [EK01]; the authors proceed to introduce the Levy process generating copula and provide lattice method to price a single name derivatives. There are results of the same flavor on studying multidimensional Levy processes and corresponding Levy copulas [CT02], [KP03], not going into lattice methods though.

We were not able, however, to identify any results on applying copulas to solve the practical problem of approximating the local dependence structure of the tree approximation of a diffusion, which allows to consider the findings presented in this note original.

References

[CT02] R. Cont and P. Tankov, *Dependence structure of spectrally positive multidimensional levy processes.*, Journal of Computational Finance **7** (2002), no. 7, 1–49.

[EK01] N. Weber E. Kellezi, *Numerical methods for levy processes: Lattice methods and the density, the subordinator and the time copula.*, 2001.

[FD01] F.Mercurio and D.Brigo, *Interest rate models. theory and practice*, Springer, 2001, Appendix C.

[HA93] J. Hull and A.White, *The pricing of options on interest rate caps and floors using the hull-white model*, The Journal of Financial Engineering (1993), no. 2, 287–396.

[KP03] J. Kallsen and P.Tankov, *Characterization of dependence of multidimensional levy processes using levy copulas.*, 2003, `http://people.math.jussieu.fr/~tankov/levycopula.pdf`.

[PMS02] P.Embrechts, A. McNeil, and D. Straumann, *Correlation and dependence in risk management: properties and pitfalls.*, Risk Management: Value at Risk and Beyond (M.A.H. Dempster, ed.), Cambridge University Press, 2002, pp. 176–223.

[Sch99] P. Schoenbucher, *A tree implementation of a credit spread model for credit derivatives*, 1999, University of Bonn, Working paper, `http://papers.ssrn.com/sol3/papers.cfm?abstract_id=240868`.