SoChi: a local moment surface pricing method of the basket credit products.

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Abstract

We propose a bottom-up dynamic credit modelling framework. To achieve a non-trivial coupling, the marginal survival probability processes are multiplied by a common exponential martingale process. Still being a factor coupling, this approach relies on convolution of the conditionally independent random variables. However, due to the much better analytical tractability, this approach allow getting rid of the traditional recursions as convolution methods, and it does not require tuning the factor quadrature, as the factor integration step is not present. Also the model can be entirely specified only in terms of the local moment surface of the common factor process, with different moments affecting different segments of the loss distribution.

Contents

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1 Introduction

One of the main unsolved problems in the modern mathematical finance is finding a portfolio credit model, which will reproduce prices of the market tradable instruments by introducing an economically sensible though risk neutral dynamics of the underlying risks. Ever since creation of the basket credit derivatives market, the de-facto standard models, from diversity scores to genetically modified Gaussian copulas, have failed to deliver adequate results robustly.

Truly dynamic models followed soon. These came in two flavours: bottom-up and top-down ones. The former follows the natural approach of coupling the marginal default events to arrive at the cumulative portfolio loss distribution. The most notable example of such approaches are [CRT07], [Lop08], [IL07]. The main challenges for such approaches was ensuring the non-negativity of the coupled default intensity and/or ensuring the intertemporal consistency of the calibrated model parameters and their stability from day to day. All being factor models, they required the conditionally-independent convolution step, introducing the corresponding numerical issues.

The top-down approaches model the cumulative portfolio loss distribution directly, thereby initially losing the marginal information. Recently, several advances have been reported on making *random thinning* work for the top-down models (for example [HT08]), which aims specifically at reintroducing the marginal information back into the model. Even with random thinning working, it is still an open question, how to adapt the top-down approaches to pricing the bespoke tranches.

In this work we present a bottom-up approach that has properties not previously reported in the literature. Firstly, we introduce a new "recursion", which lacks well known instabilities of those usually applied at the conditionally independent step of the factor copula-based approaches. Most importantly, the approach presented allows to reduce specification of the coupling only to specification of the local moment surface of an essentially bounded exponential martingale, which introduces the coupling, without specifying this process explicitly. We show that with such coupling different sections of the loss distribution appear to depend on different moments, to a certain extent. This allows interpreting the moments as the model parameters, which allow to directly twist different areas of the loss distribution.

2 The model

2.1 The marginal default model

We start with a portfolio of defaultable assets. The default event of a particular asset occurs just once and leads to the termination of the asset. We suppose that initially the processes of the individual defaults are described in terms of default intensities

$$\lambda_i(t), \quad t \geq 0,$$

which are assumed to be reasonable (non-negative) stochastic processes adapted to a given filtration $\mathcal{F}_t$. All processes $\lambda_i(t)$ $i = 1, \dots, n$ are assumed to be independent.

Consider n i.i.d. unit expectation exponential random variables ξ_i, independent of $\mathcal{F}_\infty$. The marginal default times τ_i are defined as:

$$\tau_i = inf \left\{ t : \int_0^t \lambda_i(s) ds \geq \xi_i \right\}, \tag{1}$$

which implies that the $\mathcal{F}_\infty$-conditional survival indicators will be given by the conditional expectations with respect to $\mathcal{F}_\infty$:

$$\mathbb{E}\left(1_{\{\tau_i>t\}}|\mathcal{F}_\infty\right) = S_i(t) = \exp\left(-\int_0^t \lambda_i(s)ds\right).$$

Clearly, the processes $S_i(t)$ are independent, $\mathcal{F}_t$ - adapted, decreasing and positive.

The forward (conditional) survival probabilities $P_i(t,T)$, $t<T$ are defined by:

$$P_i(t,T) = \frac{1}{S_i(t)}E\left(S_i(T)|\mathcal{F}_t\right) = E\left(\exp\left(-\int_t^T \lambda_i(s)ds\right)|\mathcal{F}_t\right). \tag{2}$$

We may assume that processes $P_i(t,T)$ are well defined and analytically tractable. This is the case for the standard assumptions regarding processes $\lambda_i(t)$ and it is widely accepted in the interest rate modeling. For instance, we could assume that $\lambda_i(t)$ are general OU processes. The latter assumption, however, does not imply that $\lambda_i(t)>0$ precisely, but at least with sufficiently high probability.

2.2 Multiplicative coupling through a common exponential martingale

Now we introduce a coupling of the marginal default indicators. The coupled indicators will be denoted $\widetilde{1}_{\{\tau_i>t\}}$. We start with the independent processes $S_i(t)$ introduced above, which possess the properties of a single name survival probability. The relation between the coupled, conditional on t survival indicator $\widetilde{1}_{\{\tau_i>t\}}$ and the process $S_i(t)$ is now different and is defined as follows.

Definition 1 ("SoChi" coupling) *The $\mathcal{F}_\infty$ conditional survival indicators*

$$\mathbb{E}\left(\widetilde{1}_{\{\tau_i>t\}}|\mathcal{F}_\infty\right) = \widetilde{\mathcal{S}}_i(t) = S_i(t)\mathcal{E}(t), \tag{3}$$

where $\mathcal{E}(t)$ is an $\mathcal{F}_t$-adapted process with the following properties:

1. $\mathcal{E}(t)\geq 0$
2. $\mathcal{E}(0)=1$
3. *$\mathcal{E}(t)$ is a martingale, uniformly integrable on each finite interval.*
4. *$\mathcal{E}(t)$ is independent of the processes $\lambda_i(t)$ and of the default generating exponentially distributed variables ξ_i.*

A straightforward calculation shows that the assumption above is, under certain technical conditions, equivalent to the following transformation of the default times:

$$\widetilde{\tau}_i = inf\left\{t : \int_0^t \lambda_i(s)ds - \log\left(\mathcal{E}(t)\right) \geq \xi_i\right\}. \tag{4}$$

Introducing a generalized intensity $\nu(t)$ by

$$\nu(t) = -\frac{\partial}{\partial t}\log\left(\mathcal{E}(t)\right),$$

we obtain that the proposed transformation of the survival probabilities is (in some at least quasi-rigorous sense) equivalent to a certain intensity transformation of the jump process that drives default:

$$\acute{\lambda}(t) = \lambda(t) + \nu(t).$$

This form for a particular case of $\mathcal{E}(t)$ is considered in several papers, (e.g. [CRT07]). We omit the question, in which sense the generalized intensity exists, especially when $\mathcal{E}(t)$ is a diffusion, or even totally discontinuous jump process. Evidently, it does not exist in any straightforward sense. Moreover, when transforming the intensity, we face the question whether the resulting "intensity" $\acute{\lambda}(t)$ is positive, provided that $\lambda(t)$ is. For example, we can not expect that $\nu(t)$ is positive, since we require that $\mathcal{E}(t)$ is a martingale. Note here that this kind of problems was often ignored in many of the related papers.

To avoid dealing with non-negativity of the intensity, we suppose to deal with the survival curve martingale correction $\mathcal{E}(t)$ directly instead of constructing the transformation of the intensity. The crucial property of the proposed transformation is that the transformed curves $\widetilde{\mathcal{S}}_i(t)$ generate exactly the same conditional survival probabilities $P_i(t,T)$ as initial ones:

$$\frac{1}{\widetilde{\mathcal{S}}_i(t)} \mathbb{E}\left(\widetilde{\mathcal{S}}_i(T)|\mathcal{F}_t\right) = \frac{1}{S_i(t)\mathcal{E}(t)} \left(S_i(T)\mathcal{E}(T)|\mathcal{F}_t\right) = P_i(t,T),$$

which follows from independence of all the processes and the fact that $\mathcal{E}(t)$ is martingale. A practical implication of this fact is that this coupling reproduces not only the individual spot survival curves, but also the *forward* survival curves. When individual name forward curves are calibrated accordingly we may then calibrate parameters of the common factor process $\mathcal{E}(t)$ to match certain correlation-related market information, like FTD, NTD or CDO prices.

3 Application: pricing of the correlation-dependent products

The main advantage of SoChi coupling is seen when the time-t default and loss/recovery distributions are constructed, i.e. distributions of variables

$$D_t = \sum_{i=1}^{n} 1_{\{\tau_i \le t\}}, \tag{5}$$

$$L_t = \sum_{i=1}^{n} N_i(\tau_i)\left(1 - R_i(\tau_i)\right) 1_{\{\tau_i \le t\}}, \tag{6}$$

$$R_t = \sum_{i=1}^{n} N_i(\tau_i) R_i(\tau_i) 1_{\{\tau_i \le t\}}. \tag{7}$$

Note that (despite it is often missed in the literature) in general it is fundamental to have $N_i(\tau_i)$ and $R_i(\tau_i)$ to be defined as explicit functions of the default time. This is necessary to make the portfolio loss/recovery distributions consistent with those of the single names. The point is that even if either notionals and recoveries are just time dependent, being evaluated at random time they become stochastic themselves. Therefore just making notionals and recoveries time dependent, let alone making them stochastic, will require a much more numerically involved convolution for construction of the loss/recovery distributions than the one used to convolve the integer scaled Bernoulli variables (which the famous recursion from [LAB03] is).

Any numerical implementation of losses and recoveries convolution to come up with the portfolio ones (6) and (7) will require approximating the products $R_i(\tau_i)N_i(\tau_i)$ and $(1 - R_i(\tau_i)) \times N_i(\tau_i)$ by integer multiples of what is come to be known as loss and recovery units u_l and u_r correspondingly. We have to note that there are approaches in the literature that try to approximate loss and recovery distributions directly via numerical schemes that nay only resemble convolution visually, not being convolutions, however. We stick with convolution as it clearly separates approximation of the random variable from application of the accurate numerical method. That is, we do choose to approximate single name loss and recoveries as integer multiples of the corresponding units, but after that we merely convolve the bounded integer valued conditionally independent random variables.

As we elaborate in the following sections, the main point of introducing SoChi coupling is to reduce specification of the dependency structure to specification of only the first several moments of $\mathcal{E}(t)$, for several fixed t at most, i.e. entirely in terms of the **local moment surface**. Furthermore, we show that for a given time different segments of the loss distribution (i.e. different values of L_t) will depend on different moments, allowing to locally affect the loss distribution via the model parameters. As a consequence, it should be possible, in principle, to calibrate the local moment surface using some kind of a bootstrapping procedure.

To implement convolutions we will be using the probability generating function (pgf) approach. A pgf is a specially constructed generating function, mostly useful for dealing with integer-valued random variables (as opposed to characteristic and moment generating functions, useful for general or non-negative continuous random variables). For an integer-valued random variable L, its pgf $\phi(z)$ is defined as

$$\phi(z) = \mathbb{E}(z^L) = \sum_i \Pr(L = i) z^i.$$

This is a power series. If random variable is bounded then the above can be written as

$$\phi(z) = z^{-L_{\min}} Q(z),$$

where $Q(z)$ is polynomial. If L is non-negative and bounded then pfg is merely polynomial with the $i-$th power coefficient equalling $\Pr(L = i)$.

As with any generating functions, pgf are most useful for convolving the independent random variables. In the case of the integer-valued independent variables L_1 and L_2, we obtain for the pgf of their sum:

$$\phi(z) = \mathbb{E}\left(z^{L_1+L_2}\right) = \mathbb{E}\left(z^{L_1}\right)\mathbb{E}\left(z^{L_2}\right) = \phi_1(z)\,\phi_2(z)\,.$$

Thus pgf of the sum is still a power series, or just a polynomial if $\phi_1(z)$ and $\phi_2(z)$ are polynomials. Therefore the problem of convolving non-negative bounded integer-valued random variables reduces (exactly) to multiplying of polynomials. This can be done either coordinatewise (using "long multiplication", aka "recursion") or using FFT (see Chapter 30 of [CCLR01] for comparative analysis and background). In the latter case, FFT is supposed to proved the exact result of convolution, as opposed to the situation when it is used to numerically convolve continuos random variables.

3.1 Motivation: default distribution of the uniform portfolio

Consider first the case of the portfolio where all marginal survival probabilities are same for all times and are not stochastic; denote them all $S(t)$. The common factor process is therefore the only source of the stochastic dynamics. This is a very common limiting case, often considered in the top-down approaches (for example, [HW08]).

In what follows we assume that time t is fixed. Denote $m(t,k) = \mathbb{E}\left(\mathcal{E}(t)^k\right)$, the $k-th$ moment of $\mathcal{E}(t)$.

Proposition 2 *For a basket of* $\boldsymbol{n}$ *names, coupled with (3),* $\Pr(D_t = i), i = 0, \ldots, n$ *depends only on the moments* $m(t, n-i), m(t, n-i+1), \ldots, m(t,n)$.

Proof. Conditional on $\mathcal{E}(t)$, the cumulative number of defaults follows binomial distribution with success probability $1 - \mathcal{E}(t)S(t)$, therefore

$$\begin{aligned}\Pr\left(D_t = i | \mathcal{E}(t)\right) &= \binom{n}{i}\left(1 - \mathcal{E}(t)S(t)\right)^i \left(\mathcal{E}(t)S(t)\right)^{n-i} \\ &= \binom{n}{i}\sum_{k=0}^{i}(-1)^k\binom{i}{k}\left(\mathcal{E}(t)S(t)\right)^k\left(\mathcal{E}(t)S(t)\right)^{n-i} \\ &= \sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{n}{i}\mathcal{E}(t)^{n+k-i}S(t)^{n+k-i}.\end{aligned} \tag{8}$$

Now taking expectation with respect to the measure of $\mathcal{E}(t)$ obtain

$$\begin{aligned}\Pr\left(D_t = i\right) &= \mathbb{E}^{\mathcal{E}(t)}\left(\sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{n}{i}\mathcal{E}(t)^{n+k-i}S(t)^{n+k-i}\right) \\ &= \sum_{k=0}^{i}(-1)^k\binom{i}{k}\binom{n}{i}m(t, n+k-i)S(t)^{n+k-i},\end{aligned} \tag{9}$$

which does depend only on $m(t, n-i), m(t, n-i+1), \ldots, m(t,n)$. ■

Although trivial, **Proposition 2** illustrates the key ingredients and implications of SoChi. In constructing loss/recovery distributions, we can first obtain $\Pr\left(D_t = i|\mathcal{E}(t)\right)$, convolving somehow the conditional on $\mathcal{E}(t)$ probability generating functions of the individual losses/ recoveries. This corresponds to the conditional convolution step of the factor copula-based approach (using FFT, as in [LG05], or by coordinate-wise multiplication, as in [LAB03]). However, as illustrated by (9), we then will be able to integrate the conditional on $\mathcal{E}(t)$ default distribution analytically, using the fact that all conditional probabilities $\Pr\left(D_t = i|\mathcal{E}(t)\right)$ depend on some powers of $\mathcal{E}(t)$ multiplicatively.

Most importantly, **Proposition 2** already shows the localizing property of SoChi. In the factor copula approach, the copula affects the loss distribution as a whole. It has proven quite challenging to come up with a copula that would affect particular segments of the loss distribution. The only approaches that were partly successful in this respect required introducing some sorts of "jumps" in the common factor, either explicitly (as in [TNCV04]), or by distorting the distribution function of the factor (as in [Xu06]). In both cases however, the most affected area of the loss distribution can be identified only vaguely, as either rather "senior" tranche, or rather "mezzanine" tranche, etc. As **Proposition 2** illustrates, localization in SoChi is much more explicit. For example, the full survival probability $\Pr(L_t = 0)$ depends only on the highest moment $m(t,k)$ (independently of the loss amounts).

3.2 Fixed loss and recovery distributions

Here we restrict our consideration to the case of notionals and recoveries that do not depend on time, i.e. $N_i(t) = N_i$, $R_i(t) = R_i$. Generalization to the case of conditionally independent notionals and recoveries is technically straightforward, however its practical feasibility under SoChi will strongly depend on the dynamics of the common factor. We leave this case as a future research subject.

Since loss and recovery distributions are constructed in exactly the same way only changing the unit and the integer multiples, we study the general case. Assume that the payoff on default of $i-$th name is integer l_i. Then, conditional on $\mathcal{E}(t)$, the marginal pgf of the contingent on default payoff (loss or recovery) will be

$$\phi_i\left(z|\mathcal{E}(t)\right) = \mathbb{E}\left(z^{l_i 1_{\{\tau_i \le t\}}}|\mathcal{E}(t)\right) = \mathcal{E}(t)S_i(t) + z^{l_i}\left(1 - \mathcal{E}(t)S_i(t)\right). \tag{10}$$

This is a polynomial of the degree l_i in z, and it is linear in $\mathcal{E}(t)$.

Theorem 3 *In addition to the assumptions of* ***Proposition 2****, assume that the payoff on default is l_i, such that the maximum payoff is $L^{\max} = \sum_i l_i$.. Then the cumulative payoff distribution $\Pr(L_t = i) = q_i$, $i = 0, \ldots, L^{\max}$, will be given by*

$$q_i = \sum_{k=1}^{n} m(t,k)\pi_{k,i},$$

where $m(t,k) = \mathbb{E}\left(\mathcal{E}(t)^k\right)$, the $k-th$ moment of $\mathcal{E}(t)$, and $\pi_{k,i} = \pi^n_{k,i}$ is the $n-$th step of the two-dimensional recursion, defined as

1. *Initialization step $\pi^0_{0,0} = 1$; $\pi_{k,j} = 0$ if $k > 0$ or $j > 0$.*
2. *For $s = 1, \ldots, L^{\max}$*

$$\pi^{s+1}_{k,j} = S_i(t)\left(\pi^s_{k-1,j} - \pi^s_{k-1,j-l_i}\right) + \pi^s_{k,j-l_i}, \tag{11}$$

where we assumed that

$$\pi^s_{k,j} = 0, \text{ if } k < 0 \text{ or } j < 0. \tag{12}$$

Proof. Substituting for notional simplicity, $\mathcal{E}(t) = x$, the conditional on $\mathcal{E}(t)$ cumulative portfolio loss probability generating function is given by

$$\Phi\left(z|\mathcal{E}(t)\right) = \prod_{i=1}^{n}\left[xS_i(t) + z^{l_i}\left(1 - xS_i(t)\right)\right]. \tag{13}$$

Assume that we have already multiplied first s marginal polynomials. Due to commutativity of multiplication the order of multiplication clearly does not matter, i.e. without loss of generality, these could have been the polynomials, corresponding with the assets with indices $1, \ldots, s$. Denoting $L^s = \sum_{i=1}^{s} l_i$,the product of such multiplication will be the polynomial

$$\sum_{k=0}^{s}\sum_{j=0}^{L^s}\pi^s_{k,j}x^k z^j.$$

Keeping the multiplication order, multiplying the above with the $(s+1)-$st multiple from (13) yields:

$$
\begin{aligned}
&\sum_{k=0}^{s}\sum_{j=0}^{L^s}\pi^s_{k,j}x^k z^j\left[xS_i(t)+z^{l_i}-xz^{l_i}S_i(t)\right]\\
&=\sum_{k=0}^{s}\sum_{j=0}^{L^s}\pi^s_{k,j}x^{k+1}z^jS_i(t)+\pi^s_{k,j}x^kz^{j+l_i}-\pi^s_{k,j}x^{k+1}z^{j+l_i}S_i(t)\\
&=\sum_{k=0}^{s+1}\sum_{j=0}^{L^{s+1}}\pi^s_{k-1,j}x^kz^jS_i(t)+\pi^s_{k,j-l_i}x^kz^j-\pi^s_{k-1,j-l_i}x^kz^jS_i(t)\\
&=\sum_{k=0}^{s+1}\sum_{j=0}^{L^{s+1}}\left[S_i(t)\left(\pi^s_{k-1,j}-\pi^s_{k-1,j-l_i}\right)+\pi^s_{k,j-l_i}\right]x^kz^j=\sum_{k=0}^{s+1}\sum_{j=0}^{L^{s+1}}\pi^{s+1}_{k,j}x^kz^j,
\end{aligned}
$$

yielding (11).

After the $n-$th step, recall that $\mathcal{E}(t)=x$, therefore the conditional cumulative portfolio pgf will be given by

$$\Phi\left(z|\mathcal{E}(t)\right)=\sum_{k=0}^{n}\sum_{j=0}^{L^{\max}}\pi^n_{k,j}\mathcal{E}(t)^kz^j. \tag{14}$$

Now taking the expectation of the above with respect to the measure of $\mathcal{E}(t)$ yields

$$\Phi\left(z\right)=\mathbb{E}^{\mathcal{E}(t)}\left(\sum_{k=0}^{n}\sum_{j=0}^{L^{\max}}\pi^n_{k,j}\mathcal{E}(t)^kz^j\right)=\sum_{j=0}^{L^{\max}}\sum_{k=0}^{n}\pi^n_{k,j}m(t,k)z^j. \tag{15}$$

Hence,

$$\Pr(L_t=j)=\sum_{k=0}^{n}\pi^n_{k,j}m(t,k).$$

This completes the proof. ■

Once we have reduced the problem of constructing the loss/recovery distribution to polynomial multiplication, it can be implemented either coordinate-wise (which the above recursion is), or using FFT. The proof above presents the coordinate-wise multiplication.

The recursion implied by the **Theorem 3** preserves all fine properties of the one from [LAB03]. First and foremost, it implies the "forward" only dependency of the next step's coefficients on those of the previous step. The table below illustrates this dependency, assuming that k is the vertical coordinate and j is the horizontal one:

$$\begin{array}{|c|c|} \hline \text{for } k=0: & \pi^{s-1}_{0,j-l_s} \longrightarrow \pi^s_{0,j} \\ \hline \text{for } k>0, j<l_s: & \text{for } k>0, j\ge l_s: \\ \hline \begin{array}{c}\pi^{s-1}_{k-1,j}\\ \downarrow \\ \pi^s_{k,j}\end{array} & \begin{array}{ccc}\pi^{s-1}_{k-1,j-l_s} & & \pi^{s-1}_{k-1,j}\\ & \searrow & \downarrow \\ \pi^{s-1}_{k,j-l_s} & \longrightarrow & \pi^s_{k,j}\end{array} \\ \hline \end{array} \tag{16}$$

The immediate practical consequence of (16) is that, similarly to [LAB03], on the $s-th$ step one needs only to compute the coefficients $\pi^s_{k,j}$ only for $k \le s$ and $j \le \sum_{i=1}^{s} l_i$, where the assets are indexed by the order in which they are added to the recursion.

Recursion (11) can be done in place, similarly to [LAB03] , however, contrary to it, recursion (11) is "destructive". Having started with all elements in the matrix $\pi^0_{k,j}$ equalling to zero, except the upper left $\pi^0_{0,0} = 1$, the recursion gradually pushes the non-zero elements to the lower-right corner, overwriting some of the non-zero elements with zeros, as prescribed by the boundary condition (12). For example, already after the first step, $\pi^1_{0,0} = 0$. For the uniform loss amount portfolio, the resulting matrix $\pi^s_{k,j}$ will be "triangular" with non zero elements only below and on the secondary diagonal, which is another way of formulating **Proposition 2**.

Forward dependency in j also implies (as it does for the recursion in [LAB03]) that the recursion does not have to be continued for all j, and it can be stopped at any given $j \le L^{\max}$. This may save considerable amount of computation in pricing CDO tranches with sufficiently low detachment points.

Finally, diagram (16) illustrates that the recursion is to be implemented first row-wise and then column-wise. On the $i-$th step, it should start with the element $\pi_{i,L_i^{\max}}$, with $L_i^{\max} = \sum_{i=1}^{s} l_i$, and proceed to the left along the row and then from right to left for the previous row etc. As we have already noted, the boundary conditions, nullifying certain close to the boundary matrix elements will have to be implemented literally as written in (12).

The most important implication of **Theorem 3** is that one only needs to specify the local moment surface $m(t, k)$ of to come up with the loss distribution; knowledge of $\mathcal{E}(t)$ itself is not needed. We find this somewhat equivalent to pricing in terms of the local vol surface in other derivatives asset classes. Localization in terms of time is obvious. In the following section we show that the moments influence on the different segments of the loss distribution is also at least partly localized in SoChi. It is these two results that allows us to speak of SoChi as a local moment surface pricing method.

3.3 Localization

In this section, **Proposition 2** is generalized to the general case of arbitrary loss amounts and survival probabilities. We start by developing some intuition for the final result, based on the uniform portfolio case. Conditional on $\mathcal{E}(t)$, the pgf of the cumulative portfolio loss in uniform case is given by

$$\Phi\left(z|\mathcal{E}(t)\right) = \prod_{i=1}^{n}\left[\mathcal{E}(t)S(t) + z\left(1-\mathcal{E}(t)S(t)\right)\right] \qquad (17)$$
$$= \left[\mathcal{E}(t)S(t) + z\left(1-\mathcal{E}(t)S(t)\right)\right]^{n}.$$

This yield loss distribution (8) immediately, as the $i-th$ power of z coefficient is given by the binomial formula.

This is a special case of (13). The only two differences between (8) and (17) are different $S(t)$ and different powers of z corresponding to different assets. Because of that we cannot use the binomial formula and have to employ recursion to obtain the $i-th$ power coefficient. If only survival probabilities were different, but all $l_i = 1$, then we would obtain immediately that

$$\Pr\left(D_t = i|\mathcal{E}(t)\right) = \sum_{\substack{(k_1,\ldots,k_n)\\ \in\sigma(1,\ldots,n)}}\prod_{s=1}^{i}\left(1-\mathcal{E}(t)S_{k_s}(t)\right)\prod_{t=i+1}^{n}\left(\mathcal{E}(t)S_{k_s}(t)\right) \qquad (18)$$

$$= \mathcal{E}(t)^{n-i}\sum_{\substack{(k_1,\ldots,k_n)\\ \in\sigma(1,\ldots,n)}}\prod_{s=1}^{i}\left(1-\mathcal{E}(t)S_{k_s}(t)\right)\prod_{r=i+1}^{n}S_{k_r}(t), \qquad (19)$$

implying that $\mathcal{E}(t)$ comes into this expression in power no less than $n-i$. This means that **Proposition 2** holds for the case of arbitrary survival probabilities, but constant loss amounts.

Proposition 4 *Under the assumptions of* ***Theorem 3****, for the $k-th$ moment $m(k,t)$, the lowest portfolio loss state $L^k_{\min}$, whose probability depends on $m(k,t)$, is the least sum of loss amounts of arbitrarily selected $(n-k)$ assets; in other words*

$$L^k_{\min} = \min_{1\le s_1<\ldots<s_{n-k}\le n}\left\{\sum_{j=1}^{n-k} l_{s_j}\right\}.$$

Proof. Rewriting (13) as the products of binomials in $\mathcal{E}(t)$

$$\Phi\left(z|\mathcal{E}(t)\right) = \prod_{i=1}^{n}\left[\mathcal{E}(t)S_i(t)\left(1-z^{l_i}\right)+z^{l_i}\right],$$

and collecting terms for each power $\mathcal{E}(t)^i$ yields:

$$\Phi\left(z|\mathcal{E}(t)\right) = \sum_{i=0}^{n}\mathcal{E}(t)^i\sum_{\substack{(k_1,\ldots,k_n)\\ \in\sigma(1,\ldots,n)}}\prod_{s=1}^{i}\left(S_{k_s}(t)\left(1-z^{l_{k_s}}\right)\right)\prod_{r=i+1}^{n}z^{l_{k_r}}. \qquad (20)$$

The portfolio cumulative loss states, probability of which will depend on $\mathcal{E}(t)^i$, are therefore given by all possible combinations of powers of z for the fixed i. As implied by (20), for each i, such powers will be produced by multiplying i terms $1-z^{l_{k_s}}$ and $(n-i)$ terms $z^{l_{k_r}}$, for all possible permutations of the asset indexing vector $(k_1, ..., k_n)$. However, denoting $\widetilde{L} = \sum_{s=1}^{i} l_{k_s}$, observe that

$$\prod_{s=1}^{i} \left(1 - z^{l_{k_s}}\right) \prod_{r=i+1}^{n} z^{l_{k_r}} = \left(1 + a_1 z + \ldots + a_{\widetilde{L}} z^{\widetilde{L}}\right) \prod_{r=i+1}^{n} z^{l_{k_r}},$$

for some coefficients a_j, some of which can be zero. This is the polynomial of degree not less than $\sum_{s=i+1}^{n} l_{k_s}$, the sum of the powers of the terms in the remaining product. Since (20) contains summation by all permutations of the indexing vector, the lowest degree of the product will be given by the least sum of $n - i$ loss amounts. ■

3.4 Sensitivity calculation

Fast calculation of the credit sensitivities is possible because all $\Pr(L_t = j)$ are linear in all $S_i(t)$. This can be seen by writing down explicitly the last step in the recursion, combined with multiplication of the final coefficient matrix by the vector of moments $m(t, k)$:

$$\Pr(L_t = j | S_i(t)) = \sum_{k=0}^{n} m(t, k) \left[S_i(t) \left(\pi_{k-1,j}^{n-1} - \pi_{k-1,j-l_i}^{n-1}\right) + \pi_{k,j-l_i}^{n-1}\right]. \tag{21}$$

This implies that its derivative with respect to $S_i(t)$ does not depend on $S_i(t)$:

$$\frac{\partial}{\partial S_i(t)} \Pr(L_t = j | S_i(t)) = \sum_{k=0}^{n} m(t, k) \left(\pi_{k-1,j}^{n-1} - \pi_{k-1,j-l_i}^{n-1}\right), \tag{22}$$

implying that, similarly to [LAB03], SoChi allows the fast sensitivity calculation with arbitrarily large jump in the marginal survival probability.

To construct the derivative by $S_i(t)$ we only need to obtain $\pi_{k,j}^{n-1}$. Those can be reconstructed from the last step matrix $\pi_{k,j}^{n}$ by inverting the last iteration recursion, using the forward only dependency. Specifically, assume that $i-$th asset had loss amount l_i. Then the inversion algorithm is

1. for the top (zeroth) row, $k = 0$: $\quad \pi_{0,j}^{n-1} = \pi_{0,j+l_i}^{n}$,
2. for the following rows,
   if $k < n$ and $j \leq L^{\max} - l_i$: $\quad \pi_{k,j}^{n-1} = \pi_{k,j+l_i}^{n} - S_i(t) \left(\pi_{k-1,j+l_i}^{n-1} - \pi_{k-1,j}^{n-1}\right)$,
3. for $k = 1$ or $j \geq L^{\max} - l_i$: $\quad \pi_{k,j}^{n-1} = 0$.

This will reconstruct all relevant $\pi_{k,j}^{n-1}$, yielding the gradient (22), knowing which the bumped probabilities of all states will be given by

$$\Pr(L_t = j | S_i(t) + \Delta S_i(t)) = \Pr(L_t = j | S_i(t)) + \frac{\partial}{\partial S_i(t)} \Pr(L_t = j | S_i(t)) \Delta S_i(t). \tag{23}$$

Note that this does not imply redoing the full recursion, replacing the removed assets by the one with probability $S_i(t) + \Delta S_i(t)$. Once $\pi_{k,j}^{n-1}$ are reproduced (which we assume to have been computed only up to the tranche's detachment point to begin with), the bumped probabilities will be immediately available by (23).

As it was mentioned in [Chi08], sensitivity calculation for the factor copula is most stable, when it is computed not for loss distribution, but already for the expected loss of a tranche, which is linked the second cumulative function of loss $H^2(L_t)$. This defined as

$$H^2(x) = \sum_{i=0}^{x} \sum_{i=0}^{x} \Pr(L_t = i) = \sum_{i=0}^{x} (x - i + 1) \Pr(L_t = i). \quad (24)$$

It is straightforward to show, that the sensitivity calculation approach presented here can be reformulated for the scheme, based on H^2, too. The follows from the fact that

$$\begin{aligned} H^2(x) &= \sum_{j=0}^{x} (x - j + 1) \Pr(L_t = j) \\ &= \sum_{i=0}^{x} (x - j + 1) \sum_{k=0}^{n} m(t, k) \left[S_i(t) \left(\pi_{k-1,j}^{n-1} - \pi_{k-1,j-l_i}^{n-1} \right) + \pi_{k,j-l_i}^{n-1} \right] \\ &= \sum_{k=0}^{n} m(t, k) \sum_{i=0}^{x} (x - j + 1) \left[S_i(t) \left(\pi_{k-1,j}^{n-1} - \pi_{k-1,j-l_i}^{n-1} \right) + \pi_{k,j-l_i}^{n-1} \right]. \end{aligned}$$

It is not clear however, whether implementing sensitivities based on H^2 is worthwhile for SoChi. The problem in the copula setup was due to the necessity to compute the conditional expected loss in the tail of the factor distribution, where the conditional default/survival probabilities were becoming too close to 0 or 1. This is not to be expected in SoChi, as there are no conditional default/survival probabilities to be computed. Therefore, while calculations through the H^2 are possible, they may not be adding any value for (on average) doubling the calculation time.

3.5 A more general setup

If we lift the assumption that $N_i(t) = N_i,\ R_i(t) = R_i$, then the situation does not get much more complicated theoretically, but it becomes considerably more involved practically. Consider the following schematic setup. For the fixed t, consider a time grid of m notes, defined as

$$0 = t_0 < t_1 < \ldots < t_m = t.$$

Assume that for the $i-th$ asset, conditioned on on $\tau_i \in (t_{k-1}, t_k],\ k = 1, \ldots, m,$ the loss $l_{i,k}$ amount is still an integer random variable (a multiple of some loss unit) taking values $[l_{i,ks}]$, $s = 1, \ldots, n^{i,k}$ and its conditional pfg is given by a polynomial

$$\xi_{ik}(z) = \mathbb{E}\left(z^{l_{i,k}} | \tau_i \in (t_{k-1}, t_k] \right) = \sum_{s=1}^{n^{i,ks}} p_{i,ks} z^{l_{i,ks}}.$$

We also assume that conditioned on the vector of the default times, all $l_{i,k}$ are independent. Then the pfg of the cumulative loss will be given by

$$\Phi\left(z|\mathcal{E}(t)\right) = \prod_{i=1}^{n} \left[\mathcal{E}(t_m) S_i(t_m) + \sum_{k=1}^{m} \left(\mathcal{E}(t_k) S_i(t_k) - \mathcal{E}(t_{k-1}) S_i(t_{k-1}) \right) \sum_{s=1}^{n^{i,k}} p_{i,ks} z^{l_{i,ks}} \right]. \quad (25)$$

Now, denoting $\mathcal{E}(t_m) = z_m$, we note that (25) still a polynomial in $z, z_1, \ldots, z_m$. Clearly, such polynomials can be multiplied, and then powers of $z_i, \ldots, z_m$ can be substituted for the corresponding joint moments of $\mathcal{E}(t_1), \ldots, \mathcal{E}(t_m)$. The latter observation can complicate the matters considerably.

We therefore restrict our consideration to the case when $\mathcal{E}(t)$ has independent increments. Under this assumption, consider the most general form of (25) for the fixed number of time points. Denoting

$$\begin{aligned} \Delta\mathcal{E}_0 = & \quad 1 \\ \Delta\mathcal{E}_k = & \quad \mathcal{E}(t_k) - \mathcal{E}(t_{k-1}), \quad k > 0 \end{aligned}$$

we obtain for the portfolio loss distribution pgf

$$\Phi\left(z|\Delta\mathcal{E}_0, \ldots, \Delta\mathcal{E}_m\right) = \prod_{i=1}^{n}\sum_{k=0}^{m}\sum_{s=0}^{n^{i,k}} \alpha_{i,is}\Delta\mathcal{E}_k z^{l_{i,ks}}. \tag{26}$$

This form is more convenient for multiplication if we put $z_k = \Delta\mathcal{E}_k$, because

$$E\left(\prod_{i=1}^{r}\Delta\mathcal{E}_{k_i}\right) = \prod_{i=1}^{r}E\left(\Delta\mathcal{E}_{k_i}\right), 1 \leq k_{i_1} \leq \ldots \leq k_{i_r} \leq m$$

There are two aspects that will determine the feasibility of a particular numerical scheme for (25). The first one is $n^{i,k}$, which is mainly deal-specific and is only partly model-specific. Indeed, notional amortization will typically come from the deal specification, however the number of the recovery states will almost surely come from the stochastic recovery model. Notionals can also be made stochastic, but this only alters the highest power of z of the marginal polynomial.

At the same time, the number of the time discretization points m increases the dimensionality of the convolution, which is truly bad. Even if notionals amortize non-stochastically, but in continuous time, the time discretization grid can be quite dense, which may dramatically increase the dimensionality of the convolution.

In principle, there are three ways to implement the multiplication in (26). It will always be possible to write down a brute force "recursion", implementing the coordinate-wise multiplication. Clearly, this will be the most inefficient method. With FFT, if m is small, then one could use an $(m+1)-$dimensional FFT, but this is clearly restricted only for $m < 3$.

Another alternative is to introduce the lexicographical order for the coefficients of the marginal polynomials

$$\sum_{k=0}^{m}\sum_{s=0}^{n^{i,k}} \alpha_{i,is}\Delta\mathcal{E}_k z^{l_{i,ks}} = \sum_{i=0}^{mn^{i,k}} \beta_i u^t,$$

effectively rewriting the marginal multivariate polynomial as a univariate one; (Section 6.6 in [Rot03] provides a good introduction into the lexicographic orders).

With this reformulation, the full procedure to construct the unconditional loss distribution would be

1. use the one-dimensional FFT to compute (26),

2. reconstruct the basis monomials from their order in the product,

3. substitute the terms $\prod_{i=1}^{r} E\left(\Delta\mathcal{E}_{k_i}\right)$ with the products of the moments of the corresponding orders,

4. collect coefficients for each power of z to arrive at the loss distribution.

Sensitivity calculation is still doable by implementing the polynomial division of the cached result of the step 1.

This procedure is quite involved, especially if m is large. A possible remedy to the latter situation is constructing the common process in such a way that it remains constant over most time intervals. This will mean that, independently on the underlying time discretization, which will still be conditioned by the marginal loss/recovery model or deal specification, the basket model dynamics will only depend of much smaller number of random variables $\mathcal{E}(t)$ for t belonging only to a subset of $t_{0,\ldots,}t_m$. In the limiting case, one can introduce only one random variable $\mathcal{E}(T)$, where T is deal maturity, effectively reducing SoChi to a static copula model on top of a possibly dynamic marginal conditional survival probabilities.

4 General construction of the common factor process

The most general way to construct the common process is to make it a stochastic exponent of some base martingale process μ. That is $\mathcal{E}(t)$ might be defined as the unique solution of the stochastic equation

$$d\mathcal{E}(t) = \mathcal{E}(t-)d\mu,\ \mathcal{E}(0) = 1.$$

To emphasize the base martingale μ we will use the notation $\mathcal{E}(\mu, t)$ in this section.

Example 5 *A simple compensated counting process*

$$\mu(t) = \Delta 1_{\{\tau \le t\}} - \Delta\lambda t \wedge \tau,$$

where τ is an exponentially (with expectation λ) distributed stopping time and $\Delta > -1$ is the value of the jump at moment τ.

The stochastic exponent, corresponding to μ is defined by

$$\mathcal{E}(\mu, t) = \exp\left(\ln(1+\Delta)1_{\{\tau \le t\}} - \Delta\lambda t \wedge \tau\right),$$

and the k-th moment $m(t,k)$ is equal to

$$\mathbb{E}\left(\mathcal{E}(\mu,t)^k\right) = (1+\Delta)^k \frac{1 - e^{-\lambda(\Delta k+1)t}}{(\Delta k + 1)} + e^{-\lambda(\Delta k+1)t}.$$

Example 6 *Compensated Poisson process*

$$\mu(t) = \Delta N_t - \Delta\lambda t,$$

where N_t is Poisson process with intensity λ and $\Delta > -1$ is a parameter defining the jump value.

The stochastic exponent, corresponding to μ is

$$\mathcal{E}(\mu, t) = \exp\left(\ln(1+\Delta)N_t - \Delta\lambda t\right),$$

and the k-th moment $m(t,k)$ is equal to

$$\mathbb{E}\left(\mathcal{E}(\mu,t)^k\right) = \exp\left(\lambda t\left((1+\Delta)^k - 1 - \Delta k\right)\right).$$

A different way to construct $\mathcal{E}(t)$ makes use of the cumulant functions of semimartingale processes. In a general setting, the basic fact is that for a special semimartingale process ξ_t the process

$$\mathcal{E}(t) = \exp\left(z(\xi_t - \xi_0) - \int_0^t G(z,s)ds\right)$$

is a local martingale. The process $G(z,s)$ is the cumulant of ξ and may be expressed in terms of the corresponding predictable characteristics.

For example, assuming that the process ξ_t is a one-dimensional, locally infinite-divisible, Markovian process, satisfying certain technical conditions on jumps, we have

$$G(s,z) = \frac{1}{2}a(s,\xi_s)z^2 + b(s,\xi_s)z + \int\left(e^{zy} - 1 - zy\right)\Lambda(\xi_s, dy)$$

where $a(s,x) > 0$, $b(s,x)$ and $\Lambda(\xi_s, t, dy)$ denote diffusion, and local Levy measure of the process respectively. The standard source of the analytically tractable examples is the case when the cumulant G is non-random, but probably time-dependent function. In this case the structure of the process ξ is well known,

$$G(s,z) = \frac{1}{2}a(s)z^2 + b(s)z + \int\left(e^{zy} - 1 - zy\right)\Lambda(s, dy).$$

As a rule a stationary distribution of jumps is assumed and thus

$$\Lambda(s,dy) = \lambda(s)\pi(dg).$$

The resulting (local) martingale $\mathcal{E}(t)$ is equal to

$$\exp\left(z\int_0^t \sqrt{a(s)}dW_s + zJ_t - \frac{z^2}{2}\int_0^t a(s)ds - \int_0^t ds\int\left(e^{zy} - 1\right)\Lambda(s,dy)\right),$$

where W_t is a standard Wiener process and J_t denotes a compound Poisson process, corresponding to the Levy measure $\Lambda(s,dy)$. The k-th moment $m(t,k)$ of the martingale process may be computed explicitly, using the martingale property and the fact that the cumulant is non-random:

$$\begin{aligned}\mathbb{E}\left(\mathcal{E}(\mu,t)^k\right) &= \exp\left(\int_0^t \left(G(s,kz) - kG(s,z)\right)ds\right)\\ &= \exp\left(\frac{z^2(k^2-k)}{2}\int_0^t a(s)ds + \int_0^t (e^{kzy} - ke^{zy} + (k-1))\Lambda(s,dy)ds\right).\end{aligned}$$

Note that z, $a(s)$, $\Lambda(s,dy)$ - represent a set of parameters which seems to be rich enough to calibrate the model to several market prices and maturities.

Since $\mathcal{E}(t)P_i(t)$ is a probability, it must be bounded by 1 from above, which implies certain restrictions on the process $\mathcal{E}(t)$. Specifically, we need to assume that the essential supremum of the random variable $\mathcal{E}(t)$ is finite:

$$\mathsf{esssup}\,(\mathcal{E}(t)) = ||\mathcal{E}(t)||_\infty = \lim_{k\to+\infty} \sqrt[k]{m(t,k)} < \infty. \quad (27)$$

This shows immediately that the diffusion term in the latter example of an exponential martingale should be omitted and that $zy < 0$, $\Lambda(s, dy)$ - almost everywhere. In other words the process zJ_t should have negative jumps and positive compensator.

5 Further research

The approach presented here is at its infancy, so a lot needs to be done to make it work in practice. On the theoretical side, specification of the factor process $\mathcal{E}(t)$ needs to be narrowed down, such that it calibrates to the market observables. A link between pricing the market observable tranches and the bespoke ones needs to be established. Applicability to more complex payoffs than (5)-(7) needs to be evaluated, in particular, practical feasibility of the scheme needs to be researched.

One special case is most interesting, in our opinion. When the marginal survival curves are not stochastic, it is fairly easy to create $\mathcal{E}(t)$, which would evolve only in discrete time, thus making SoChi more feasible for the payoffs that need the general setup (26). It this setup, the upper bound on $\mathcal{E}(t)$ can be determined for all relevant times, which simplifies construction of the discrete time $\mathcal{E}(t)$ that will immediately satisfy (27). This would be the natural way of extending SoChi to a stochastic recovery setup under the market standard assumption on the non-stochasticity of the credit curves.

Since SoChi will almost surely require heavy calculations, numerically optimized schemes to implement (27) must be researched as well. As we are dealing with the polynomial multiplication, it is reasonable to implement it using the "divide and conquer" idiom, which in this case will naturally lead to a parallel implementation. We therefore expect highly parallelizable frameworks like CUDA to appear very useful.

Finally, since SoChi reduces specification of the coupling to specification of the local moment surface, it looks worthwhile to investigate a purely non-parametric approach of calibrating the surface directly. If the upper boundary of the factor process can be established, than calibration will essentially be reduced to solving a moment problem for a bounded exponential martingales. The moment sequences for the bounded random variables are well characterized since [Sch91], however generalizing this result to the exponential martingales is, to our knowledge, still an open problem.

References

[CCLR01] T. H. Cormen, S. Clifford, C. E. Leiserson, and R. L. Rivest, *Introduction to algorithms*, 2nd revised ed., The MIT Press, 2001.

[Chi08] A. Chirikhin, *On CDO tranche pricing when copula is nearly comonotone*, 2008, http://www.chirikhin.com/docs/ComonotoneCopulaPricing.pdf.

[CRT07] Alexander Chapovsky, Andrew Rennie, and Pedro A. C. Tavares, *Stochastic intensity modelling for structured credit exotics*, CREDIT CORRELATION: Life After Copulas (Alex Lipton and Andrew Rennie, eds.), World Scientific Publishing Company, 2007.

[HT08] I. Halperin and P. Tomecek, *Climbing down from the top: single name dynamics in credit top down models*, 2008, http://www.defaultrisk.com/pp_model191.htm.

[HW08] John Hull and Alan White, *Dynamic models of portfolio credit risk: A simplified approach*, Journal of Derivatives **15** (Summer 2008), no. 4, 9–28.

[IL07] S. Inglis and A. Lipton, *Factor models for credit correlation*, 2007, http://www.defaultrisk.com/pp_corr106.htm.

[LAB03] J. Sidenius L. Andersen and S. Basu, *All your hedges in one basket*, Risk (2003), 67–72.

[LG05] J.-P. Laurent and J. Gregory, *Basket default swaps, CDOs and factor copulas*, Journal of Risk **7** (2005), no. 4, 103–122.

[Lop08] Andrei V. Lopatin, *A simple dynamic model for pricing and hedging heterogenous CDOs*, 2008, http://www.defaultrisk.com/pp_cdo_68.htm.

[Rot03] J. Rotman, *Advanced modern algebra*, 2nd ed., Prentice Hall, 2003.

[Sch91] K. Schmudgen, *The k-moment problem for compact semi-algebraic sets*, Mathematische Annalen (1991), no. 289, 203–206.

[TNCV04] Pedro A.C. Tavares, Thu-Uyen Nguyen, Alexander Chapovsky, and Igor Vaysburd, *Composite basket model*, "Derivatives Week" (2004), http://www.defaultrisk.com/pp_crdrv_48.htm.

[Xu06] Geng Xu, *Extending gaussian copula with jumps to match correlation smile*, 2006, http://www.defaultrisk.com/pp_corr100.htm.