Reflecting Firm Value model and DVA/CVA

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Abstract

Having appreciated that the credit component of an uncollateralized derivative represents *new* borrowing and hence that it is as much a derivative as a straight bond, we propose an extension of a Merton-type Firm Value Model to deal with valuation of such component (DVA/CVA) in a risk-neutral setting as an alternative to a reduced form approach. This allows obtaining several economically sensible properties of DVA/CVA not easily or directly obtainable with the reduced form approach, and also shows the limits of applicability of several assumptions behind the reduced form approach.

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Version 1.0: 21 April 2014
Current version 1.1: 12 May 2014

1 Introduction

CVA/DVA and related subjects (collectively branded as xVA hereon) have clearly become the problem of the decade. Different parts of xVA have been introduced by numerous authors over past 15 years: Levy and Clarke (2000); Hull and White (2001), Brigo and Masetti (2005), Alavian et al. (2008), Gregory (2009), Castagna (2011), Burgard and Kjaer (2012) or Bielecki and Rutkowski (2013) to name a few. But compared to similar subjects of the previous decades, the diversity of approaches dealing with xVA in general, and with its numerous nuances that keep popping up, is truly unprecedented.

Shall it be priced under risk neutral or historical measure, as required by BaselIII (2010)? Is Henry-Labordère (2012) a true alternative to American Monte Carlo (e.g. Cesari et al. (2009)) as the general production approach? How many repeated AMCs are necessary when collateral switch is to be priced in, implementing approaches like Piterbarg (2012) or Pallavicini et al. (2012)? Does it always require a major IT infrastructure to be implemented in production or a handheld may suffice (Andreasen (2014)), perhaps backed up by a GPU personal supercomputer (xCelerit (2012))? And more practically, how to deal with ever changing compositions of the netting sets, how to compactify the seemingly very big data, how (often) to (re)calibrate the mega pricing model, once it is constructed so as to resolve the tradeoff between reproducing prices on sufficient variety of instruments and and still being reasonably fast, how to speed up calculation of greeks and, last but not least, how to stabilize theta (assuming all other components of P&L prediction are working perfectly)? The universe is clearly expanding, while entropy decrease is not obvious.

In the grand scheme of things our work is rather inertial. We do not endeavor to propose either a front office production quality model, or a mathematische annale. Our goal is to present a rather simple but holistic framework incorporating both pricing traditional debt and the credit component of the collateralized contingent claims. Despite the simplicity of our approach, it allows justifying or refuting in a fairly straightforward ways several issues common to the mainstream xVA pricing approaches. As such, we hope the approach presented will be of the pedagogical value at least.

Specifically, we firstly acknowledge that the credit component of an unsecured derivative represents *new* borrowing with stochastic notional. Therefore valuation of this component (i.e. CVA/DVA) must be approached as a pricing new debt. Motivated by the treatment of derivatives in accounting, we then develop a version of a Reflecting Firm Value approach to deal with pricing of xVA in an arbitrage free way by extending the traditional firm value approach. We discuss the intuition and the benefits of using such approach in dealing with xVA valuation, compared to a reduced form framework. Our approach allows us to observe various interesting effects, which one should expect from a generic xVA model, e.g. contributing change in DVA to the PNL, general incorrectness of assuming independence between the default of the firm and the future value of its derivative portfolio, and relevance of the all netting sets in pricing DVA/CVA on each of them.

Certain aspects of our approach somewhat overlap with those in Section 3 of Hull and White (2013). The main difference is that the firm value approach allows obtaining results directly without introducing sensitivity of default intensity to leverage. A classical structural approach to CVA was considered in Lipton and Savescu (2013). A simple model, conceptually similar to ours was used in Section 4 of Burgard and Kjaer (2011), however to address a different problem; Section 4 below derives a similar result in our setting.

2 The case for a Firm Value model

Risk-neutral pricing is an extremely attractive valuation methodology, as its implementation amounts to computing expectations of discounted payoffs under the equivalent martingale measure. Once the model has been calibrated, the great variety of numerical methods exist to produce the pricer. Model risk can further be quantified by stressing the user provided model parameters, which are initially assumed constant.

The method is actually a shortcut for finding the present value of the strategy replicating the contingent claim being priced. Pricing by replication is justified by the fact that the market is arbitrage free by assumption. Uniqueness of the arbitrage price and equality of such price to the one given by the risk neutral valuation method is established for a complete market.

Definition of an arbitrage-free market starts with specifying the underlying tradeables that can then be used to construct replication strategies. In certain versions of this construction, the tradeables are allowed to at most disappear (e.g. in the case of default) or otherwise change in a predictable way, e.g. shares can be diluted when warrants are exercised, which needs to be taken into consideration when such warrants are priced. It is assumed however that new underlying risks are not created.

It is clear that the liability created by a contingent claim is *new* to the firm that accepts the liability. Consider a firm that is unlevered at time 0, i.e. who's assets are initially financed entirely by equity. The balance sheet equation for such firm is:

$$A_0 = E_0.$$

Assume that for some reason the firm has engaged into a transaction with a credit risk free entity, the transaction creating a pure liability to deliver a contingent claim at time $t > 0$. This includes the case of a straight borrowing, when the firm promises to repay a fixed amount at a given time. More generally, assuming that the fair value of such claim is V_0, that the firm has received cash $C_0 = V_0$ for entering such transaction and that the liability is tracked at market value in the balance sheet, its equation becomes

$$A_0 + C_0 = E_0 + V_0.$$

Thus the firm has become leveraged and V_0 represents the value of the new liability. As this general case contains issuing a straight bond, pricing the credit component of a general contingent claim should be somewhat similar to that of pricing a straight new bond.

Pricing new bonds is a subject of Corporate Finance. Typically, approaches used here are not based on arbitrage, simply because arbitrage cannot be justified economically, especially in case of plain borrowings. When you invest in a new bond he do not transfer risk that already exists in the economy; you take a new risk. You can perfectly hedge it only by selling. The risk stems from the fact that one cannot know perfectly how the firm is going to use the newly raised cash.

It is clear that the firm's credit spreads prior to issuance of the new debt do not provide information about the firm's post-leveraging credit worthiness. The problem here is that a CDS, being a delta one credit derivative, merely swaps an *existing* credit risky bond for an *existing* credit risk free bond. As with any derivative, existing risks are only sliced and transferred, but no new risks are created.

In the limiting case considered above, where the firm was not leveraged initially, the firm could not default at all before it was leveraged as there was nothing to default on. (It could of course go out of business having lost all shareholders' money, but this would not be a default). Thus the initial credit spread was zero. This situation (and a rather close situation when some non-public debt is available) poses the biggest challenge to CVA desks that have to rely on absolutely subjective proxying, which is, essentially, a limiting case of a corpfin approach (pricing based on "comparables").

It is difficult to adapt a reduced form/intensity based model to price new liabilities. One construction is described in Hull and White (2013), which relied on certain sensitivity coefficients. Such coefficients if constants, or parameters of the coefficients' functional form would have to be estimated from the co-movements of the firm's credit spreads and its leverage, which can only be obtained from the accounting statements. Thus one has to employ some kind of a structural argument, at least at the model estimation stage. To justify the overall approach, one would have to explain how the newly issued bonds could have been hedged by existing bonds, which, in case of bonds, does not make sense, because an investor in the new bonds would be taking new risk, represented by the firm's activities being financed by the new issue. Such investor would not be seeking to replicate this new risk by investing into the existing bonds.

An alternative to the reduced form framework is the Firm Value Model (FVM). The key assumption here is that the firm's assets are tradeable and the firm equity and liabilities are effectively derivatives on the firm's assets. Under the tradeability assumption, the arbitrage pricing framework does apply. The replication argument here is that the investor in the debt is actually investing in the firm's asset mix and he should be indifferent between investing via the firm's bonds and investing into the assets directly by replicating the bond's value synthetically. This assumption is, of course, the Achilles hill of the model, as the fair market value of the assets is not observable.

Economically, one can argue that a firm's default is always driven by deterioration of the *liquidation* value of at least some of the firm's assets. In the case of missed coupons the firm's management would find it impossible (or impractical) to liquidate some assets at the prevailing market prices to get cash necessary to pay the coupon. This means that the true liquidation value of the assets has fallen (below some barrier). In the case of inability to refinance, the refinancing investors would not be interested in investing synthetically into the firm's assets also for the reasons that the amount of debt is higher than the liquidation value of the assets. Inability to refinance will drive the default; this also includes the case of inability to borrow a little for the short term against some of the assets to cover just the coupon payments.

A Firm Value approach is initially not designed to deal with new liabilities. It assumes that the firm already has a zero bond and the problem is only to value it given the dynamics of the assets. It is however relatively straightforward to adapt the model to pricing new liabilities by assuming that, in case of straight borrowing, all cash raised is immediately invested in some known assets. Equivalently, one can think that the newly issued bonds are immediately paid in the asset and not in cash. The new assets mix defines the new asset dynamics and then the rest of the model applies.

Our ultimate goal is to price xVA corrections to the otherwise fair value of derivatives. As opposed to the case of pricing straight new bonds, the asset which is being financed by the new liability is often known. There are two extreme cases: a dealer that creates derivatives by exercising the replicating hedging strategy and an investor, which can ultimately be a hedger or a speculator. In the dealer's case, the potential liability is invested in the strategy replicating the liability, so the additional asset is known. In case of a hedger, no new assets are presumably invested in as the derivative is used to hedge existing risks. The case of a speculator is the most difficult one, however it is the least interesting practically, because obviously speculating counterparties will most likely be dealt with only on the collateralized basis. The point here is that despite the liability that may be created is new, there may be much less uncertainty about the marginal asset then there would be in case of a standard setting for the Firm Value model (and, actually, for any marginal credit model).

Thus there are several opportunities.

1. Ignore the fact that we are dealing with a new borrowing and simply treat xVA like adjustments as values of CDSs with stochastic notional. Some parts of wrong way risk may be introduced as part of the underlying factor dynamics. This approach is the easiest to implement, but it is largely mechanistic, as there is no other justification to it in general except for simplicity. Arguably, this is better than nothing. Despite being the most widely used in practice, there is very little to analyze in this approach.

2. Tweak a reduced form/intensity based approach to deal with new borrowings. This will be largely a phenomenological model, and it will require utilizing the structural arguments and financial statement analysis to calibrate/estimate. Inability to justify pricing by replication will require some equilibrium arguments. This actually may work in practice, but it is less analytically tractable.

3. Embrace the Firm Value approach. Justification of the risk neutral valuation is not totally fair, as full hedging strategy is still not fully implementable. The method is the easiest to adapt to pricing marginal liability embedded into uncollateralized derivatives, but dynamics of the the other asset liquidation values remains obscure and impossible to calibrate. The method is thus not much less mechanistic than either 1 or 2. However, as we show below, this approach allows deriving and justifying several interesting features of xVA, which are otherwise debatable and not obvious. Being the hardest to implement in practice in its most simple form described below, this approach may still be useful to frame the mainstream discussion of xVA by identifying constraints and explaining the limiting cases.

Our work will be focusing on option 3, thus complementing the mainstream research in the area of xVA, which mainly focuses on options 1 and 2.

3 A single period Reflecting Firm Value model

3.1 The Merton's Model

The assumptions in this and the following sections are as follows

1. The market is complete and arbitrage free, and money market account is the numeraire.

2. Firm assets are tradeable on the market

3. The risk free rate is zero. We will comment where relevant how this assumption can be relaxed without loss of generality.

Consider a firm that starts operating at time t with the shareholder investment of C_0. At the very same moment the shareholders choose to leverage the firm. The firm is then instantaneously leveraged by borrowing the notional amount of N and the market value D_0. The total proceeds are then used to instantaneously purchase the initial assets with market value A_0, such that the the mark to market balance sheet equation holds

$$A_0 = C_0 + D_0.$$

The question is, given the above assumptions and the risk-neutral dynamics of A_t, what will be the dynamics of the market values of the shareholder equity C_t and debt D_t for $t > 0$. It is clear that, at any moment of time, the fair balance sheet total equation should hold

$$A_t = C_t + D_t. \tag{3.1}$$

Remark 1. For definiteness, we assume that all of A_t, C_t and D_t are positive in this context.

The most basic "Merton type" firm value model (FVM) (see Black and Scholes (1973); Merton (1973)) is aimed at modeling a possible default of the firm and it is specified as follows.

1. The whole borrowing is represented by a single zero coupon bond with notional N and maturity T.
2. The firm's asset A follow Geometric Brownian Motion (which can be greatly relaxed up to allowing it to be) and there are not dividend payments from the assets for $t \in [0, T)$.
3. At time T,

- if $A_T > N$, then the zero bond is redeemed (or refinanced, resetting the situation to $t = 0$)
- otherwise the firm defaults, as it is unable either to repay the bond notional or find a refinancing source (since it would be, in normal conditions, irrational to for the investors to finance a firm with loan to value larger than 1). Thus the default is purely mechanical.

Stochasticity of the firm assets makes default event stochastic and there are no other factors that drive the default event.

An important feature of the model is that the firm can only default at T. It cannot default before that time because there are no payments to default on: nothing is due to be paid on borrowed funds before that time. Note that the assumption that the firm's only borrowing is the zero bond, economically, means that the firms indeed does not have any other borrowed items like "accounts payable" etc; it does have only one item of borrowings and it has to make a single payment on such borrowing, the redemption payment.

If the firm does not default at time T then its shareholders have the choice of either stopping the firm operations by selling the assets, paying the bond notional N and realizing the profit $C_T - C_0$, which is necessarily positive, or refinancing. The latter case splits in many opportunities, e.g. retaining the earnings fully, effectively reinvesting them, or paying some fraction of C_T as dividends (presumably liquidating a fraction of assets to release cash), as long as at least N of present value of the new funds can be raised. Whatever happens, the second case effectively resets the problem to $t = 0$, perhaps with different initial equity. We therefore don't deal with this case and focus on determining the dynamics of for $t \in [0, T]$.

Assuming that the firm is not making any payments to either the shareholders or the bond holders until T, the bondholders would receive at most N, if $A_T > N$, the equity holders receiving the residual value of the assets, $A_T - N$. Otherwise the firm defaults and the bondholders end up with the assets $A_T < N$, while the shareholders receive nothing.

Thus the payoff to the bond holders is

$$\min(N, A_T) = N - (N - A_T)^+ .$$

Since arbitrage pricing applies by assumption, the time-t value of this payoff is exactly the time-t market value of the firm's debt, i.e.

$$D_t = N - \mathbb{E}^Q (N - A_T)^+ = N - Put(A_t, N, T).$$

Correspondingly, the payoff to the shareholders is that of a call on the firm assets, therefore

$$C_t = \mathbb{E}^Q (A_T - N)^+ = Call(A_t, N, T).$$

The balance sheet total equation (3.1) in this case is nothing more than the call put parity:

$$A_t = D_t + C_t = N - Put(A_t, N, T) + Call(A_t, N, T).$$

The above valuation formulas allows us to define general CVA/DVA, which is not part of the standard FVM nomenclature.

3.2 DVA/CVA and basic properties

Definition 2. Define Debit Value Adjustment (DVA^f), from the firm's perspective, as

$$DVA_t^f = \mathbb{E}^Q (N - A_T)^+ , \tag{3.2}$$

such that

$$D_t = N - DVA_t^f.$$

In particular, the total amount of cash borrowed at $t = 0$ is $N - DVA_0^f$.

Definition 3. Define Credit Value Adjustment (CVA), from the investor's perspective as

$$CVA_t^i = DVA_t^f.$$

Thus from anybody's perspective the fair value of investment is

$$D_t = N - DVA_t^f = N - CVA_t^i.$$

In what follows, unless stated otherwise, we will be presenting formulas from the firm's perspective and omit the "f" superscript.

Fact 4. *Necessarily,* $DVA_t > 0$. Follows immediately, as it is a put.

Plugging this into the balance sheet total equation (3.1) yields two equalities

$$A_t = C_t + N - DVA_t, \tag{3.3}$$
$$A_t + DVA_t = C_t + N. \tag{3.4}$$

Accounting standards allow the firms to track their bonds either at present value, or at book value. Equations (3.3) and (3.4) represent these two cases.

Equation (3.4) represents the classical book value accounting situation at time $t = 0$, where bonds are tracked at face value and any premium or discount at the issuance time is amortized over the lifetime of the bond. It is this case which is really the motivation behind the Merton's approach and its derivatives that rely upon the face value. Note that since we are dealing only with zero bonds, the bonds is necessarily issued with a discount, which DVA is.

The problem with this method is that if DVA_0 is subsequently not revalued, but only amortized, then equation (3.4) holds only at time $t = 0$. If the assets were tradeable, this would distort PNL, making it inconsistent with FVM.

Equation (3.3) represents the case when bonds are tracked at market value. In this case the balance sheet equation is nothing more than the call/put parity. As we will show below, in this case FVM justifies changes in DVA affecting PNL. In this case, the balance sheet total always equals the market value of assets, which makes perfect sense.

If face value accounting for debt allowed market revaluation of DVA, then the two approaches would be equivalent as equation (3.3) would hold for all t. In this case DVA_t could be interpreted as a non-performing asset which the firm must invest in due to its credit riskiness. It also can be considered a balancing account that makes the total assets equal to the equity plus debt notional, where equity is valued as a call option on A_t. Also note that if considered an asset, DVA defined above has the famous property of the shareholders' benefit on default: this is the amount the shareholders do not have to repay if the firm defaults.

Definition 5. The firm's Profit and Loss (PNL) at time t is defined as

$$PNL = \mathrm{d}C_t = \mathrm{d}A_t + \mathrm{d}DVA_t.$$

This general relationship is not strong enough by itself to rule out nonsensical co-movement of PNL and DVA. However under FVM these two components are linked by a call/put parity, hence

$$\begin{aligned} \frac{\mathrm{d}C_t}{\mathrm{d}A_t} - \frac{\mathrm{d}DVA_t}{\mathrm{d}A_t} &= 1, \\ 0 < \frac{\mathrm{d}C_t}{\mathrm{d}A_t} &< 1, \\ -1 < \frac{\mathrm{d}DVA_t}{\mathrm{d}A_t} &< 0. \end{aligned}$$

ThusPNL and DVA always move in the opposite directions, and being an option on the assets the $|\mathrm{d}DVA| < |\mathrm{d}A|$ and $|PNL| < |\mathrm{d}A|$. When assets grow, the fall in DVA is smaller, therefore positive PNL is generated. On the contrary, when assets fall, increase in DVA is never large enough enough hence negative PNL is produced. Therefore, in the latter case, even though the value of the DVA benefit to the shareholders is increasing (the shareholders will return even less money in case of default), the total value of equity is still falling, so the shareholders are worse off.

Thus allowing changes in DVA to be included in PNL is consistent with FVM, and it makes general economic sense. When default probability is remote and option corresponding to the value of equity is in the money, DVA, being the out of the money option, is small and not volatile. In this case equity is as volatile as the assets.

When assets fall closer to the face value of the bond notional and even cross it before maturity of the debt, equity value approaches zero and become less volatile, because volatility in the assets is now offset by the volatility of DVA, which is the in the money option. If assets fall further, change in DVA is still added to the PNL, but it is almost perfectly offset by the loss in the assets, keeping equity value almost constant and near zero. Thus, under FVM, contributing changes in DVA to PNL is natural.

To conclude, we reformulate the interrelationship between assets, equity and debt in terms of the credit spread.

Definition 6. Define the time-t credit spread as

$$u_t = -\frac{1}{T-t} \ln \left(1 - E^Q \left[A_T > N\right]\right).$$

Claim 7. It is not possible for the firm to generate positive PNL when credit spread is widening.

This follows from the fact that both equity and bond are synthetically long assets and that the credit spread is a non-increasing transformation of the bond's value.

3.3 Reflection and wrong way risk

While fair value accounting for own bonds is optional, derivatives must be accounted for at fair value. Derivatives held by a firm may be assets (e.g. long option), liabilities (e.g. short option) or either assets or liabilities at different moments of time (e.g. a swap with the initial value of 0). The balance sheet reflects this literally.

If we consider only a single pair of the counterparts, we will assume that all derivatives between them are nettable for xVA valuation purposes. Thus, in what follows, we speak of a single derivative, which stands for the net value of the derivative portfolio between the two counterparts being considered.

Assume that the derivative expires at the same time as the firm's zero bond we considered above and that it can be both asset and liability. Denoting the market value of such standalone derivative from the firm's perspective as V_t, the balance sheet of a credit risk-free firm would reflect it as

$$\begin{aligned} A'_t &= A_t + V_t^+, \\ L'_t &= L_t - V_t^-, \end{aligned}$$

where L is the balance sheet value of the liabilities and $(x)^- = min(x, 0)$ hence $(x)^- < 0$. That is the value of the derivative is added to either assets or liabilities, "reflecting" from the total level of assets without it. Because of this observation we dub the extension of FVM to this situation "Reflecting Firm Value" (RFV) model.

Since the level of liabilities at maturity is affected and so is dynamics of the assets, the fair value of the derivatives needs to be adjusted by DVA. It is clear that even if the initial value of the derivative is zero, as long as the derivative can represent a liability at its expiry. In any case, if the firm has already been levered by the time the derivative was entered into, DVA on the existing liabilities will be affected whether the new derivative is asset, liability or both.

As we assumed that interest rates are zero, computation the total DVA of the firm can be reduced to (3.2) by conditioning on V_T, the terminal value of the derivative. Assuming that the equivalent martingale measure provides the joint terminal probability of V_T and A_T, we can write:

$$\begin{aligned} DVA &= E^Q\left[\left(N - \left[A_T + V_T^+\right]\right)^+ |V_T > 0\right] + E^Q\left[\left(\left[N - V_T^-\right] - A_T\right)^+ |V_T < 0\right] \\ &= E^Q\left[\left(N - \left[A_T + V_T^+\right]\right)^+ |V_T > 0\right] + E^Q\left[\left(N - \left[A_T + V_T^-\right]\right)^+ |V_T < 0\right]. \end{aligned}$$

Since

$$\left(A_T + V_T^+\right) 1_{\{V_T > 0\}} + \left(A_T + V_T^-\right) 1_{\{V_T < 0\}} = A_T + V_T,$$

we obtain

$$DVA = \mathbb{E}^Q \left(N - A_T - V_T\right)^+, \tag{3.5}$$

and for survival probability

$$s_t = \Pr\left(A_T + V_T - N > 0\right). \tag{3.6}$$

The key point here is that in RFV, default event is never independent form the terminal value of the derivative. Thus there is also a kind of right/wrong way risk in DVA in RFV. This is the consequence of the fact that the credit component of the collateralized derivative is new borrowing. As such, the assumption of independence between the derivative's mark to market and the counterparty default is not consistent with RFV.

One can argue that the derivative portfolio of a given firm may be known to the market hence existing CDS spreads already account for it. They probably do for the existing derivative portfolio, which in our analysis is part of A_t. They don't account for the risk of the newly added traded, which is the subject of practical xVA valuation. Furthermore, the point is not whether they just account for it, but whether the valuation model models right/wrong way risk resulting from these derivatives adequately. It definitely does not if independence is assumed.

Apart from that, (3.5) has several important features.

1. It works for $N = 0$, i.e. for the initially unlevered firm.
2. For the initially levered firm it implies that the asset/liability derivative can simply be bundled with the rest of the assets and then such bundle can be used in the straight bond DVA pricing formula.
3. The formula gives the total firm DVA given the asset/liability mix and it does not attribute this total DVA to each individual instrument. Already in the case of the firm having just one zero bond and one derivative DVA attribution is undefined. Practically one would have to think about the marginal DVA of a new derivative.

The latter item has two consequences: since only the sum $A_T + V_T$ enters the valuation formula, it is the sum's distribution function that matters. Thus there may be "netting benefit" between the assets and the derivative in terms of DVA, including the associated right/wrong way risk. The second gives rise to the concept *counterparty interference* that we analyze in the next section.

When analyzing new derivative deals, the model risk needs to be considered. An important assumption in FVM is that dynamics of the assets is known. Adding a new transaction will alter dynamics of assets and/or liabilities. This is the most transparent case of the changes in the model parameters. As we analyze below, in some cases change in the model parameters is transparent. A general case, when cash exchange is involved, has to be approached a red flag for model becoming misspecified in the very near future, once the some true asset is purchased, or, otherwise, some existing debt is retired.

In particular, an assumption is necessary about how the deal is financed if it is asset at time 0, and how proceeds are invested, if it is a liability time 0. A natural assumption is that the new deal is always financed with debt, while proceeds are used to invest into the same asset as before. Note that necessity to assume especially how proceeds are invested is the consequence of the fact that created liability is *new*. When truly lending one would naturally like to know how the proceeds would be used (think a mortgage salesman). Us making explicit assumption about it models exactly this situation. Thinking that usage of proceeds is irrelevant will most surely result in mispricing of the deal.

There are three distinct cases:

- a dealer who replicates the derivative with a hedging strategy,
- a hedger, who wishes to swap one existing asset with another, fairly, with no cash exchange,
- anything else, which may or may not include exchange of cash.

Case 1. A dealer.

A dealer, who will create an offsetting hedging strategy with a different counterparty. As we shown below, having a perfectly offsetting strategy does not alter the default probability, but does affect recovery rate. In the limiting case when the dealer is not levered, recovery is not affected either.

An important special case is when the unadjusted deal's value is 0, i.e. it is a fairly priced swap or a forward. Since it can become a liability in the future, the deal will attract DVA. The problem is that for the deal to be executed, the marginal DVA will have to be transferred to the client, as it will be the deal's CVA from the client's perspective. It should be a cash transfer (we do not consider the case where when CVA/DVA is bundled with the deal). So cash should initially exist (denoted as B_0 below):

$$A_0 + B_0 + DVA_0 = C_0 + N_0.$$

Once the new transaction is exercised, it does not alter the balance sheet equation, but B_0 is transferred to the client as marginal DVA:

$$A_0 + DVA_0^{'} = C_0 + N_0 + V_0, \tag{3.7}$$

where $V_0 = 0$. This is consistent with the view that DVA is a non-performing asset that freezes part of the balance sheet.

Case 2. A fair swap hedger.

In a fairly priced swap, existing assets are swapped for other assets. All assets are known in this case. No cash is involved. Denoting the values of the legs as L_1 and L_2, we have (from the firm's perspective):

$$L_1 = L_2.$$

We assumed that the firm has had assets being swapped. Assuming that these assets had value L_1, hence we could have written

$$A_0 = A_0^{'} + L_1$$

the balance sheet equation post the transaction becomes

$$A_0^{'} + L_1 + L_2 + DVA' = L_1 + N_0 + C_0.$$

In other words, by swapping L_1 out the firm has accepted liability equalling to L_1, while on the asset side, a new asset with the value of L_2 is recognized.

This is equivalent to the case of a dealer that has new assets of $A_0^{'} + L_2$. Since the transaction is done by a credit sensitive firm, it will generate DVA, similarly to Case 1. As both additives are known, the marginal DVA can be computed.

Case 3. Anything else, involving cash.

If the firm does not wish to immediately sell the risk, or swap fairly an asset for another asset, including when one side of the swap is cash, then this raises questions. Where is the cash coming and where it will be invested (depending on direction). Having cash involved is not fundamentally a problem, as it is just another kind of an asset. The issue here is that cash is the most liquid asset whose purpose is to be exchanged for something else.

This case, while not materially different from Case 2, has the highest model risk. We do not deal with model risk arising from parameter instability in this work and leave it to the sequel.

3.4 Counterparty interference and the case of a dealer

Traditionally xVA research focuses on at most bilateral adjustments to the derivative values, thus considering only a single pair of counterparts. So did we until now.

In reality each firm may have derivative portfolios with many other counterparties. In practice, for a given firm individual xVA is computed for each netting set with each counterparty and each such calculation is done in isolation. This is consistent with the view that liabilities created by the net values of the derivative portfolios do not affect the firm's creditworthiness.

This is not always true in reality. If a firm accumulates a one- and wrong-sided exposure “against the street" its credit worthiness will suffer (enough tool look at the monolines' and AIG credit spreads in 2007-2009 and JP Morgan's spread in the spring of 2012).

Assume that the firm has several derivative counterparties and that the unadjusted values of the netted sets of derivatives with these counterparties is V_t^i. For notational simplicity we will assume that the firm is not leveraged otherwise, or, equivalently, that some of these derivatives are straight bonds. From default modeling perspective, all these netting sets are still nettable, since survival probability is

$$s_t = \Pr\left(A_T + \Sigma V_T^i > 0\right). \tag{3.8}$$

Note that in the above, V_T^i for the net liabilities at time T are negative, so formula contains (3.6) as a special case.

Practically, one needs to compute DVA for each netting set individually, because each counterparty needs to be quoted an individual haircut for its netting set, which is CVA from his perspective. For this (3.5) has to be altered.

Assume that at T the firm does default, i.e.

$$A_T + \Sigma V_T^i = A_T + \Sigma\left(V_T^i\right)^+ + \Sigma\left(V_T^i\right)^- < 0.$$

The total firm's DVA is still given by (3.5):

$$\begin{aligned} DVA &= \mathbb{E}^Q \left(N - A_T - \Sigma V_T^i\right)^+ \\ &= \mathbb{E}^Q \left(-\Sigma \left(V_T^i\right)^- - A_T - \Sigma \left(V_T^i\right)^+\right)^+ \\ &= \mathbb{E}^Q \left(-\Sigma \left(V_T^i\right)^- \left[1 - \frac{A_T + \Sigma \left(V_T^i\right)^+}{-\Sigma \left(V_T^i\right)^-}\right]\right)^+ \end{aligned}$$

Assuming that all debtors have equal claim on the residual assets of $A_T + \Sigma \left(V_T^i\right)^+$, we can postulate that each suffers same proportional loss of

$$\frac{A_T + \Sigma \left(V_T^i\right)^+}{-\Sigma \left(V_T^i\right)^-}. \tag{3.9}$$

Thus we can attribute DVA to the netting set by the following rule:

$$DVA_t^i = \mathbb{E}^Q \left[-\left(V_T^i\right)^- \left(1 - \frac{A_T + \Sigma \left(V_T^i\right)^+}{\Sigma \left(V_T^i\right)^-}\right)\right]^+. \tag{3.10}$$

Note that all items under the expectation operator are assumed tradeable, therefore the expectation can be taken.

An important consequence of the above is that perfect hedging of market and credit risk of the derivative portfolio, i.e. having offsetting positions but with different counterparties does not remove DVA/CVA charge in general. One can think of this as a model of a dealer that sells derivatives to clients and then hedges them with other counterparties.

The balance sheet equation in this case is

$$A_t + V_t + DVA_t = C_t + V_t + N.$$

If one counterparty is in the money, then the other will be out of the money and vice versa. If the deal's PV flips sign then the counterparties simply change sides of the balance sheet, but the equation still holds.

For this reason the perfectly hedged deal does not increase the firm's default probability by itself. Default still is driven only by A_t. However if default does happen, i.e. $A_T < N$, then full assets of $A_T + V_T$ will be used to fulfill the total obligations, equaling $N + V_T$. Having same priority, both the bond holders and the derivative holders will suffer proportional loss of 3.9. In practical terms this means that default probabilty is not affected.

Only if the firm was initially unlevered, i.e. $N = 0$, as long as A_t cannot go negative, then the firm can only default if $A_T = 0$. In this case the fully hedged dealer remains credit risk free.

The mainstream assumption that existing CDS levels can be used to price CVA/DVA and that exposure may not be correlated with the default event, actually corresponds to the case of a very lightly leveraged and fully hedged dealer. This is definitely more the case for traditional investment banks, than for commercial/investment banks, where customer deposits are present.

4 Collateralized deals and initial margin

Since we consider a one sided case, i.e. only the firm is credit risky, while the counterparty is risk free, we only need to consider the situation where the firm enters into a derivative that is liability at time 0. In the uncollateralized case, the firm would receive cash, equaling the value of the assumed liability less marginal DVA. In case of the initially unlevered firm the balance sheet dynamics is:

$$A_0 = C_0$$

$$\Downarrow$$

$$A_0 + B_0 + DVA_0 = C_0 + V_0$$

The cash will then have to be invested, and nature of the investment will affect valuation, as discussed in Section 3.3.

The collateralized deal is different in to aspects.

1. Instead of cash the firm accepts a receivable. This receivable models cash (assuming cash collateralization) that would first be given to the firm and then returned as posted collateral.

2. The notional of the receivable is equal the unadjusted value of the derivative. In the uncollateralized case, it would be less by the marginal DVA. This is intended to set the initial exposure to 0.

Thus, denoting the value of the receivable $R_0 = V_0$ we seem to have

$$A_0 + R_0 = C_0 + V_0, \tag{4.1}$$

however since receivable is nettable with the derivative we actually can consider them a "swap" like package with initially zero value:

$$\tilde{V}_0 = V_0 - R_0.$$

This package is effectively unwound at the first calculation date and replaced with a new one. Therefore a collateralized deal is equivalent to a fairly priced swap with horizon equaling the margining period.

$$A_0 = C_0 + \tilde{V}_0, \tag{4.2}$$

Contrary to the uncollateralized case, here we need not make assumption about how the proceeds would be invested. They are necessarily invested in the receivable.

As discussed in Section 3.3, such swap still generates residual DVA, because the straight derivative has stochastic value and may become liability. Therefore 4.1 and 4.2 are actually incorrect. The correct relationships are

$$A_0 + B_0 = C_0, \tag{4.3}$$

$$\Downarrow$$

$$A_0 + DVA_0 = C_0 + \tilde{V}_0, \tag{4.4}$$

where $B_0 = DVA_0$. As with any fairly prices swap, the initial value of the marginal DVA will have to be transferred to the counterparty to offset the marginal CVA. Equivalently, one can think of this transfer as posting initial margin by the credit risky firm.

5 Conclusion

We have presented an extension of Firm Value Model that naturally deals with DVA. The model explicitly demonstrates various properties of xVA that one would expect from a production model. We also provide an alternative explanation for the additional funding adjustment, which is consistent with our approach. We appropriate that the model's being very basic and literally being a Merton type Firm Value Model makes it difficult to implement in practice.

The fundamental issue here is that the model relies on risk neutral valuation in the market where a single risk free rate exists. Existence of a risk free rate is an unrealistic assumption in the today markets, both because every counterparty is risky and because most underlying products are traded on the collateralized basis. We therefore believe that proper model should start by constructing the market of collateralized assets and then construct the uncollateralized derivative synthetically. We leave presentation of such model for the sequel of this work.

References

A. Levy, M. Clarke, Modelling derivative credit risk in the presence of collateral and netting, Research Paper, JP Morgan Securities Inc. Derivatives Research. Also featured in Risk 13 (1) (2000) 100–3.

J. C. Hull, A. D. White, Valuing credit default swaps ii: Modeling default correlations, The Journal of derivatives 8 (3) (2001) 12–21.

D. Brigo, M. Masetti, Risk neutral pricing of counterparty risk.

S. Alavian, J. Ding, P. Whitehead, L. Laudicina, Counterparty valuation adjustment (cva), Available at ssrn. com.

J. Gregory, Being two-faced over counterparty credit risk, Risk 22 (2) (2009) 86–90.

A. Castagna, Funding, liquidity, credit and counterparty risk: Links and implications, Iason research paper. Available at http://iasonltd. com/resources. php.

C. Burgard, M. Kjaer, Generalised cva with funding and collateral via semi-replication, Available at SSRN 2027195.

T. R. Bielecki, M. Rutkowski, Valuation and hedging of otc contracts with funding costs, collateralization and counterparty credit risk: Part 1, arXiv preprint arXiv:1306.4733.

BaselIII, Basel iii: A global regulatory framework for more resilient banks and banking systems, Basel Committee on Banking Supervision, Basel.

P. Henry-Labordère, Counterparty credit risk-cutting cva's complexity, Risk-London 25 (7) (2012) 67.

G. Cesari, J. Aquilina, N. Charpillon, Z. Filipovic, G. Lee, I. Manda, Modelling, pricing, and hedging counterparty credit exposure: A technical guide, Springer, 2009.

V. Piterbarg, Cooking with collateral, Risk Magazine 8.

A. Pallavicini, D. Perini, D. Brigo, Funding, collateral and hedging: uncovering the mechanics and the subtleties of funding valuation adjustments, arXiv preprint arXiv:1210.3811.

J. Andreasen, Cva on an ipad mini part 1: Intro.

xCelerit, Accelerating cva computations the easy way (2012).
URL http://www.xcelerit.com/downloads/white-papers/

J. C. Hull, A. White, Collateral and credit issues in derivatives pricing, Rotman School of Management Working Paper.

A. Lipton, I. Savescu, Cdss, cva and dva - a structural approach, Risk (4) (2013) 60–65.

C. Burgard, M. Kjaer, In the balance, Available at SSRN 1785262.

F. Black, M. Scholes, The pricing of options and corporate liabilities, The journal of political economy (1973) 637–654.

R. C. Merton, Theory of rational option pricing, The Bell Journal of Economics and Management Science 4 (1) (1973) 141–183.