On CDO tranche pricing when copula is nearly comonotone.

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Abstract

Pricing of a CDO base tranche is considered when copula is comonotone and approximations are derived for the base tranche expected loss, obtained with a one-factor Gaussian copula model, when correlations are sufficiently close to 100%. Numerical examples are provided showing that a two-term approximation reproduces the true value with no more than approximately 5% relative error, uniformly across detachments and average default probabilities for correlations between 95% and 100%.

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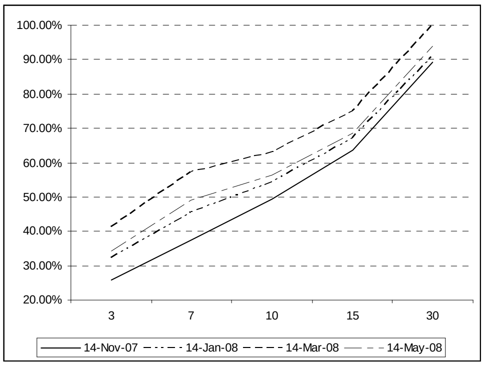
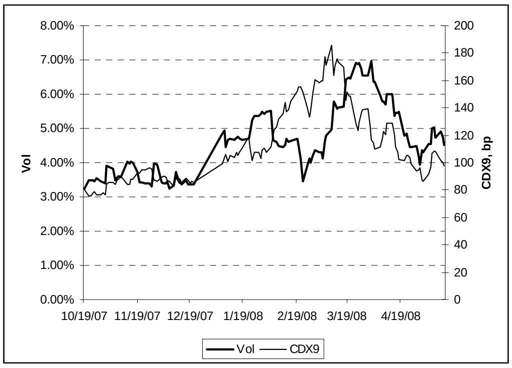
1 Introduction

The 2007-08 credit crunch has put performance and robustness of the market standard basket credit models to test. With CDS spreads generally widening due to the overall economic concerns and value moving to the more senior tranches of the CDOs, the implied Gaussian copula correlation skew moved up and flattened out. For several days the model failed to calibrate the implied correlation for the most senior CDX tranches, while on the days when it could calibrate, the implied value was close to 1. In addition to that, daily volatilities has almost doubled, increasing the credit books' exposure to CDS gamma. The charts below illustrate both implied correlation skew dynamics and realized 20 business day window daily volatility of CDX9.

CDX9 Correlation Skew



CDX9 and Its RealizedVol



Source:MarkIT

As such market conditions appear to persist, it is prudent to make sure that practically used credit basket models can operate robustly even when correlation approaches 100%, which may not have been a standard requirement before. A fundamental solution would be a complete change of the modelling approach, which is hardly achievable in practice over short periods of time. Of the true though marginal improvements to the model, perhaps, the most widely tried, is making the recovery rates stochastic on top of the existing frameworks. Unfortunately, this does not guard against implied correlations approaching 1 should market conditions worsen further. Therefore, as soon as the underlying machinery of the model may still be a nearly comonotone copula, one will have to make this machinery work robustly.

Thus, setting aside other possible changes in the model or numerical method, in this work we concentrate on improving the numerical method used to implement a nearly comonotone copula. We expect the following from such method.

1. The model should always converge if the solution theoretically exists. In February 2008, there were several days when a solution for the implied correlation would not exist at all for a market standard one-factor flat loading Gaussian copula model (hereon "Gaussian copula model"), under the assumption of a non-stochastic recovery, so other means of making the model robust were necessary. On most other days the solution would exist, although could be quite close to 1. for the most senior $0-30\%$ CDX base tranche, in which case one would expect the model to converge without a significant fall in performance.

2. Without compromising the performance so as to make the model useless, the effects of the numerical method noise should be contained, despite the obvious singularity of the method for correlations close to 1. Practically, this means minimizing the spurious CDS gamma in the portfolio. Also, if the calibration problem is well-posed, the solution should be at least continuous and should maintain that the base tranche expected loss is monotonically decreasing in base correlation and monotonically increasing in time and detachment.

In case of Gaussian copula, when a solution exists, the calibration problem is indeed well-posed. This is because when $\rho = 1$, Gaussian copula degrades into the upper Frechet-Hoeffding bound, also known as "comonotone copula". One way to interpret the comonotone copula of default times, represented by the Gaussian copula with $\rho = 1$, is that, conditioned on the copula factor, default times of the individual assets become non-stochastic and ordered.

As we show in this work, the solution to the CDO tranche problem exists in closed form in case when the default time copula is comonotone, at least for the standard CDO tranches (i.e. if the single asset recovery rates are assumed independent of the default times and single asset notionals do not depend on time). This is because the comonotone copula implied time-t tranche expected loss will still depend only on the cumulative marginal default probabilities by time t. Corresponding simple expressions for the time-t expected tranche losses are still differentiable by the marginal time-t default probabilities, allowing the comonotone case extension of the analytical CDS spread and single asset default sensitivity algorithms, which are available the factor copulas.

In what follows we first consider the Large Homogenous Portfolio case (hereon "LPH case"), which allows to build intuition about the expected tranche loss behavior in the comonotone case, and then consider the general case of the comonotone portfolio, as implied by Gaussian copula with correlation of 1 (which is a representation of the comonotone copula).

In what follows we will use the following notation:

$\mathbf{1}_{\{x\}}$	indicator function of event x
N_a	number of assets in the basket
τ_i	default time of asset $i, i = 1..N_a$ hereon
N_i, R_i	notional and recovery rate of asset i
$p_i(t)$	time-t marginal default probability of asset i
$\widetilde{p}_i(t)$	time-t conditional on the copula factor default probability of asset i
l_i, r_i	integer valued (discretized) loss and recovery amounts of asset i
u_l, u_r	loss and recovery units
L_t	cumulative portfolio loss process (time-t cumulative portfolio loss)
$EL_t(K)$	time-t expected loss of a base tranche with detachment K
$H^2(x)$	second cumulative function of a probability distribution
ρ	Gaussian copula pairwise correlation
$\Phi(x)$	standard univariate Gaussian distribution function
$\Phi^2(x, y, r)$	standard bivariate Gaussian distribution function with correlation r
$\Phi^n_r(u_1, \ldots, u_n)$	standard multivariate Gaussian distribution function with pairwise correlation r

2 Review of tranche pricing and sensitivity calculation

In this section we recall the relevant facts about tranche pricing on a portfolio of N_a assets, characterized by the cumulative default probabilities $p_i(t)$. In what follows, we will be interested in pricing a base tranche with detachment K. It can be shown (see e.g. [ASB03]) that pricing such tranches can be reduced to calculation of a series of the *time-t expected tranche losses* $EL_t(K)$, defined as

$$EL_t(K) = \mathbb{E}\left(\min(K, L_t)\right) = K - \mathbb{E}(K - L_t)^+, \tag{2.1}$$

where

$$L_t = \sum_{i=1}^{N_a} \mathbf{1}_{\{\tau_i \leq t\}} N_i(\tau_i)\left(1 - R_i(\tau_i)\right). \tag{2.2}$$

Note that the above definition of L_t is the easiest possible, i.e. even in the easiest case it is fundamental that the notional and the recovery rate of the i–th asset are evaluated at the asset's default time τ_i, hence they must be random processes, adapted to filtration, containing the information of τ_i.

The above expression can be simplified for standard CDO tranches (including the standard index tranches on CDX and iTraxx) to become:

$$L_t^{discr} = u_l \sum_i \mathbf{1}_{\{\tau_i \leq t\}} l_i, \tag{2.3}$$

where u_l is "loss unit", a nonnegative real number, and l_i are (small) non-negative integers, selected such that $\|u_l l_i - N_i(1 - R_i)\| < \varepsilon$, for some small ε and some metric $\|.\|$.

The motivation for (2.3) is to approximate the loss density $\partial_x \Pr(L_t \leq x)$, $x \in \mathbb{R}$ by the loss distribution $\Pr(L_t^{discr} = u_l k)$, $k \in \mathbb{Z}_+$, as soon as each individual loss is approximated by an integer multiple of the loss unit. Depending on both deal specification (notionals and recoveries) and modelling assumptions (mainly, stochastic vs non-stochastic recoveries and their dependency on default times), different integer approximations may be necessary.

Approximation (2.3) is rather a simple one (the simplest would be the case when $l_i = 1\ \forall i$, the "uniform" loss amount case). This approximation is possible if we assume that recovery rates are independent of default times (which is a modelling assumption), and that single asset notionals do not depend on time (which is part of the deal specification). The former assumption is inherited from the single asset credit modeling, where it is standard, while the latter is consistent with the specs of majority of the vanilla synthetic CDOs.

Specifically, as the only source of randomness in (2.3) are default times, making them conditionally independent reduces the problem of constructing $\Pr(L_t^{discr} = u_l k)$ to convolution of the integer scaled conditionally independent Bernoulli random variables. Such convolution amounts to multiplying conditional on G *probability generating functions (pgf)* $\xi_i(x|G)$ of the marginal time-t losses, which are polynomials of powers l_i.:

$$\xi_i(x|G) = E\left(x^{L_i}\right) = 1 - \widetilde{p}_i(t) + \widetilde{p}_i(t) x^{l_i}, \tag{2.4}$$

The conditional *pgf* of L_t^{discr} is then given by

$$\xi_{L_t^{discr}}(x|G) = \prod_i \xi_i(x) = \prod_i \left(1 - \widetilde{p}_i(t) + \widetilde{p}_i(t) x^{l_i}\right) \tag{2.5}$$

and, by definition of ξ, it's $k-th$ power coefficient will be $\Pr(L_t^{discr}=k|G)$:

$$\Pr(L_t^{discr}=k|G)=\frac{1}{k!}\frac{d^k}{dx^k}\ \xi_{L_t^{discr}}(x|G)|_{x=0}$$

Depending on the set of l_i for the given portfolio, multiplication of polynomials in (2.5) can be done either coordinate-wise (which is what the [ASB03] recursion method really is), or by an FFT. It is the fact that the above $\xi_i(x|G)$ has only two terms that makes the recursion method optimal, especially for the uniform loss amount case. However, if notionals were time dependent or recovery rates were stochastic (except when recoveries are comonotone only among themselves and are independent of default times), then L_i could still be discretized, but it would have more states. Hence $\xi_i(x|G)$ would contain more terms, making FFT a faster method of convolution.

Once the conditional loss distributions $\Pr(L_t^{discr}=k|G)$ are built, one can solve the original pricing problem by either first building unconditional loss distributions and integrating the payoff, or valuing the conditional payoffs and integrating it:

$$\mathbb{E}(K-L_t)^+=\sum_i(K-L_t)^+\int\Pr(L_t^{discr}=k|G)\mu(\mathrm{d}G) \tag{2.6}$$

$$=\int\sum_i(K-L_t)^+\Pr(L_t^{discr}=k|G)\mu(\mathrm{d}G) \tag{2.7}$$

Note that in all cases, they are the facts that we are merely multiplying polynomials to achieve the $\xi_{L_t^{discr}}(x|G)$ and that polynomial multiplication is associative and commutative that warrant that the "order" in which we multiply the polynomials is irrelevant. The same facts imply that probabilities of all states of L_t^{discr} are affine in $\widetilde{p}_i(t)$, yielding the famous fast CDS sensitivity algorithms, specifically

$$\frac{\partial}{\partial\widetilde{p}_i(t)}\ \xi_{L_t^{discr}}(x|G)=\frac{\xi_{L_t^{discr}}(x|G)}{(1-\widetilde{p}_i(t)+\widetilde{p}_i(t)x^{l_i})}\left(x^{l_i}-1\right)$$

$$=\xi^{-i}_{L_t^{discr}}(x|G)\left(x^{l_i}-1\right), \tag{2.8}$$

which is independent of $\widetilde{p}_i(t)$. The polynomial division in the above amounts to "undoing" one iteration of the recursion, while the last multiplication is re-doing the step for the $i-th$ asset with conditional default probability of 1. The resulting "conditional gradient" $\frac{\partial}{\partial\widetilde{p}_i(t)}\ \xi_{L_t^{discr}}(x|G)$ can be used to compute the $\xi_{L_t^{discr}}(x|G)$ for different scenarios of $\widetilde{p}_i(t)^b=\widetilde{p}_i(t)+\Delta\widetilde{p}_i(t)$, independently of the size of the perturbation.

If many scenarios per asset are to be evaluated, first computing the gradient and then computing the scenario $\xi_{L_t^{discr}}(x|G)$ by

$$\xi_{L_t^{discr}}(x|G)^b=\xi_{L_t^{discr}}(x|G)+\frac{\partial}{\partial\widetilde{p}_i(t)}\ \xi_{L_t^{discr}}(x|G)\Delta\widetilde{p}_i(t) \tag{2.9}$$

is not fundamentally faster then just caching $\xi^{-i}_{L_t^{discr}}(x|G)$ and then recomputing

$$\xi^{-i}_{L_t^{discr}}(x|G)\left(1-\widetilde{p}_i(t)^{bumped}+\widetilde{p}_i(t)^{bumped}x^{l_i}\right).$$

It is faster and more numerically stable if we use the second order conditional cumulative functions of L_t^{discr}.

Recall that given the time-t loss distribution $P_t(x) = prob(L_t \leq x)$, the property $L_t \geq 0$ implies that

$$\mathbb{E}(K - L_t)^+ = \int_0^K P_t(x)\mathrm{d}x = H^2(K), \tag{2.10}$$

where $H^2(K)$ is the "second cumulative distribution" of the random variable (see the original paper [DWS91] and [Bra04] for the follow up results). This can be proven by integration of by parts.$\mathbb{E}(K - L_t)^+$. This property is mostly known to the market practitioners in its differential form: $H^2(K)'' = P_t(K)$, i.e. the second derivative of the put on a non-negative random variable by the strike equals to the value of the "implied" density of the underlying variable.

In the discretized case, given a probability distribution $q(i)$, (2.10) specializes for $K \in \mathbb{R}$ to

$$H^2(K) = \sum_{i=0}^{\lfloor K \rfloor - 1} \sum_{j=0}^{i} q(i) + (K - \lfloor K \rfloor) \sum_{s=0}^{\lfloor K \rfloor} q(s), \tag{2.11}$$

where $\lfloor x \rfloor = \max(i \in Z_+ : i \leq x)$, and we assume that the sum for which the lower limit is larger than the upper limit equals 0. This can be proven by direct calculation and applying the discretized version of integration by parts.

Introducing a "discretized" second order cumulative function

$$h^2(k) = \sum_{i=0}^{k} \sum_{j=0}^{i} q(i), \tag{2.12}$$

the above equation can be rewritten as

$$H^2(K) = h^2(\lfloor K \rfloor - 1) + (K - \lfloor K \rfloor) \left(h^2(\lfloor K \rfloor) - h^2(\lfloor K \rfloor - 1) \right), \tag{2.13}$$

which is quite intuitive. Note that for discrete random variables, $h^2(k)$ is usually taken as the definition of second cumulative distribution function in the literature. However we stick with our definition of $H^2(x)$ to be consistent both for integer and real x.

Now observe that (2.6) can be rewritten as

$$\mathbb{E}(K - L_t)^+ = \int H^2(K|G)\mu(\mathrm{d}G)$$

However, (2.11) being linear in all $q(i)$ and the fact that all $q(i)$, which are $\Pr(L_t^{discr} = k|G)$ in our case, and are affine in each $\widetilde{p}_i(t)$ imply that $H^2(K|G)$ are affine in each $\widetilde{p}_i(t)$, therefore (2.9) also holds for the whole $H^2(K|G)$:

$$H^2(K|G)^b = H^2(K|G) + \frac{\partial}{\partial \widetilde{p}_i(t)} H^2(K|G) \Delta \widetilde{p}_i(t) \tag{2.14}$$

The implication is that once we know $\frac{\partial}{\partial \widetilde{p}_i(t)} H^2(K|G)$ (a single number, depending on K), we can compute all kinds of $H^2(K|G)^b$ for different $\Delta \widetilde{p}_i(t)$ We never need to re-do the last step of recursion with different $\widetilde{p}_i(t)^b$.

The last missing component is $\frac{\partial}{\partial \widetilde{p}_i(t)} H^2(K|G)$, It can be proven that $\frac{\partial}{\partial \widetilde{p}_i(t)} h^2(K|G)$, which $\frac{\partial}{\partial \widetilde{p}_i(t)} H^2(K|G)$ is linear function of, is given by an expression similar to (2.8):

$$\frac{\partial}{\partial \widetilde{p}_i(t)}\, h^2(k|G) = \frac{h^2(k|G)}{(1-\widetilde{p}_i(t)+\widetilde{p}_i(t)x^{l_i})}\left(x^{l_i}-1\right).$$

The proofs using the recursion formalism can be found in [DWS91] and [Bra04]. It can also be shown that algorithms involving $h^2(K|G)$ are more stable than those for $\Pr(L_t^{discr}=k|G)$, because $h^2(K|G)$ is non-decreasing function of K.

It is possible to construct $h^2(K|G)$ directly, using same recursion as for $\Pr(L_t^{discr} = u_l k)$, only with different initial step (i.e. the "empty" loss distribution $(0,\ldots,0,1)$, must be substituted by "empty" h^2, which is $(...,3,2,1)$, the right-most state being $L_t^{discr}=0$). However, it is easy to see that for the fixed K, it is approximately twice faster to first build loss distribution, and only then to "integrate" it twice to arrive at h^2. This is precisely because the initial loss distribution has zeros in the tail, while all entries of the initial h^2 are non-zero.

Thus to price a tranche it is necessary and sufficient to build the function $H^2(K)$, conditional or unconditional, depending on the needs. Note that should a CDO tranche be exposed to recovery writedown (e.g. if it is a X-100% base tranche or certain kind of a forward starting tranche), all above arguments hold for the cumulative portfolio recovery $R_t=\sum_i \mathbf{1}_{\{\tau_i\le t\}} N_i(\tau_i)R_i(\tau_i)$, although in general one will need to discretize recoveries differently, introducing a recovery unit u_r.

3 The comonotone copula

The comonotone copula (also known as upper Hoefding-Frechet bound) is given by [Nel06]

$$C^+(u_1,\ldots,u_n)=\min(u_1,\ldots,u_n).$$

As opposed to the lower Hoefding-Frechet bound, it is a copula in all dimensions.

A one-factor Gaussian copula is defined as

$$C_\rho(u_1,\ldots,u_n)=\Phi_\rho^n(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n)).$$

Using a technique, similar to the one we employ in the proof of our **Theorem** 5.1 below, it can be shown that Gaussian copula becomes comonotone copula when $\rho=1$. In what follows, we will use this fact, as representation of the comonotone copula as Gaussian copula with $\rho=1$ allows to simplify our proof.

4 The comonotone LHP case

In this section we will omit the subscript t and assume that we are interested in computing $H^2(K)$ for a given K. We will assume, without loss of generality, that K is expressed as a percentage of the full notional and that recovery rates on all assets equal zero.

Consider a LPH with the default probability p, coupled with a one-factor Gaussian copula with pairwise flat correlation ρ. In this case the LHP loss distribution is given by (see, e.g.

[Vas87])

$$P(L_t \leq K) = \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(K) - \Phi^{-1}(p)\right)\right). \tag{4.1}$$

This implies that

$$H^2(K) = \int_0^K \Phi\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}\Phi^{-1}(q) - \Phi^{-1}(p)\right)\right) \mathrm{d}q \tag{4.2}$$

$$= \Phi^2\left(-\Phi^{-1}(p), \Phi^{-1}(K), -\sqrt{1-\rho}\right), \tag{4.3}$$

This follows from the standard integral

$$\frac{1}{\sqrt{2\pi}} \int_0^T \Phi(at+b)\exp\left(-t^2/2\right) \mathbf{d}t = \Phi^2\left(\frac{b}{\sqrt{1+a^2}}, T, -\frac{a}{\sqrt{1+a^2}}\right),$$

which can be verified by differentiating by T.

H^2 is clearly continuous when $\rho \to 1-0$, as in this case $r \to -0$. Since $\Phi^2(x, y, 0) = \Phi(x)\Phi(y)$ we obtain our final result that when $\rho \to 1$,

$$H^2(K) \to K(1-p) \Longrightarrow$$
$$EL_t(K) \to Kp. \tag{4.4}$$

This is a very intuitive result. Observe that it reproduces the boundary cases: $EL_t(0) = 0$ and $EL_t(1) \to p$.

The main point is that if all default probabilities are same, as required by LHP, then the full comonotonicity implies that all default times are same too, i.e.$\tau_i = \tau, \forall i$. Thus, all paths of L_t will be just

$$L_t = \theta\left(\mathbf{1}_{\{\tau<t\}}\right), \tag{4.5}$$

where τ is the default time of the whole portfolio, and $prob(\tau < t) = p_t$, due to the full comonotonicity. Knowing (4.5), one can take expectation (2.10) directly to obtain (4.4).

5 The general comonotone case

In this section we consider a general case of the portfolio, coupled with comonotone copula.

Theorem 5.1 *Suppose that the individual default times are coupled with comonotone copula. For a given t, assume that the individual assets are ordered in the decreasing order by their time-t default probability, i.e.*

$$\forall i = 1..N_a - 1 : p_i(t) \geq p_{i+1}(t). \tag{5.1}$$

and, given this ordering, define

$$I_K = \min\{j : \sum_{i=1}^{j} l_i \geq K\}.$$

Then,

$$EL_t(K) = K - \mathbb{E}(K - L_t)^+ = u_l \sum_{i<I_K} l_i p_i(t) + p_{I_K}(K - L^{I_K - 1}). \tag{5.2}$$

Proof. We will proceed by considering (2.6) as a special case of Gaussian copula with $\rho = 1$. Given ordering (5.1), define the auxiliary "cumulative" realized loss (as the function of the index i) as

$$L^i = u_l \sum_{k \le i} l_k$$

If several assets have same $p_i(t)$ we can treat them as one with the loss amount being the sum of the loss amounts of such assets. Equivalently, we can order them among themselves in an arbitrary way without changing the further reasoning.

Given the time t, for each i define

$$G_i = \Phi^{-1}\left(p_i(t)\right).$$

Since $p_i()$ are non-decreasing functions onto $[0,1]$ such G_i will exist for all i. Note that τ_i are increasing functions of G because

$$\tau_i = p_i^{-1}\left(\Phi\left(Z_i\sqrt{1-\rho} + \sqrt{\rho}G\right)\right) \tag{5.3}$$

$$\overset{\rho=1}{=} p_i^{-1}\left(\Phi\left(G\right)\right). \tag{5.4}$$

Note that ordering (5.1) implies that

$$G_{i+1} \le G_i, \tag{5.5}$$

which implies for $G \in (G_{i+1}, G_i]$:

$$\begin{aligned}\tau_{i+1} = p_{i+1}^{-1}(\Phi(G)) > p_{i+1}^{-1}(\Phi(G_{i+1})) = p_{i+1}^{-1}(\Phi(\Phi^{-1}\left(p_{i+1}(t)\right)) = t \implies \\ \mathbf{1}_{\{\tau_{i+1} \le t | G\}} = 0,\end{aligned} \tag{5.6}$$

$$\begin{aligned}\tau_i = p_i^{-1}(\Phi(G)) \le p_i^{-1}(\Phi(G_i)) = p_i^{-1}(\Phi(\Phi^{-1}\left(p_i(t)\right)) = t \implies \\ \mathbf{1}_{\{\tau_i \le t | G\}} = 1.\end{aligned} \tag{5.7}$$

By definition of G_i's, (5.6) will hold $\forall k > i$, and (5.7) will hold $\forall k \le i$, therefore

$$\begin{aligned}L_t(G) &= u_l \sum_{k=1}^{i} l_k \mathbf{1}_{\{\tau \le t | G\}} = L^i, \text{ and} \\ prob(L_t = L^i) &= prob(G_{i+1} \le G < G_i) = \Phi(G_i) - \Phi(G_{i+1}) = \\ &= p_i(t) - p_{i+1}(t).\end{aligned} \tag{5.8}$$

Importantly, the last equation is not dependent on G, so it is, in fact, the unconditional loss distribution, and all that remains is applying (2.6).Recalling that, by definition, $L^i - L^{i-1} = u_l l_i$, for $i > 0$ and $L^0 = 0$ obtain:

$$
\begin{aligned}
\mathbb{E}(K-L_t)^+ &= \sum_i (K-L^i)^+ prob(L_t=L^i) = \sum_{0\le i<I_K} (K-L^i) prob(L_t=L^i) = \\
&= K \sum_{0\le i<I_K} prob(L_t=L^i) - \sum_{0<i<I_K} L^i prob(L_t=L^i) = \\
&= K \sum_{0\le i<I_K} (p_i(t)-p_{i+1}(t)) - \sum_{0<i<I_K} L^i (p_i(t)-p_{i+1}(t)) = \\
&= K(1-p_{I_K}) - \left(L^1 p_1(t) + \sum_{2<i<I_K} p_i(t)\left(L^i - L^{i-1}\right) - p_{I_K} L^{I_K-1} \right) = \\
&= K(1-p_{I_K}) - \sum_{i<I_K} l_i p_i(t) + p_{I_K} L^{I_K-1} = \\
&= K - \left(\sum_{i<I_K} l_i p_i(t) + p_{I_K}(K - L^{I_K-1}) \right),
\end{aligned} \tag{5.9}
$$

which, together with (2.1), implies (5.2). ■

An intuition behind (5.8) is as follows. Consider integrating the conditional expected loss

$$
\begin{aligned}
\mathbb{E}(K-L_t|G)^+ = \int \mathbb{E}(K - u_l \sum_i \mathbf{1}_{\{\tau_i \le t\}} l_i | G)^+ = \\
= (K - u_l \sum_i \mathbf{1}_{\{p_i^{-1}(G)\le t\}} l_i)^+ = L(G).
\end{aligned} \tag{5.10}
$$

going through G from $+\infty$ to $-\infty$. Clearly, when $G \to +\infty$, $L_t(G) = 0$, as is implied by (5.6). Now making G smaller we will be getting more and more default indicators in $L_t(G)$ becoming 1, and the order of them becoming one will be given by (5.5). Importantly, since $p_i^{-1}(G)$ are monotone, once the indicator become 1 (when G crosses next G_i), it always stays 1, i.e. the cumulative loss L_t is monotonically increasing function of G.

The second term in (5.2) gives the contribution of the I_K-th asset to the expected loss on the base tranche in case $L^{I_K-1} < K < L^{I_K}$. Note that if we let $K \to L^{I_K}$, then

$$
EL_t(K) \to u_l \sum_{i<I_K} l_i p_i(t) + p_{I_K}(L^{I_K} - L^{I_K-1}) = p_{I_K} u_l l_{I_k} = u_l \sum_{i\le I_K} l_i p_i(t),
$$

as one would expect.

Also note that (5.2) elementary reduces to (4.4) if $N_a \to \infty$ and K is expressed as the percentage of the pool notional.

If the $p_i(t)$ were ordered independently of t (in other words, the curves $p_i(t)$ did not cross for all t), then the intuition for (5.2) is elementary, if one thinks in terms of default times. Consider computing $EL_t(K)$ using a Monte Carlo simulation that generates default times. In this case, G there will be only one source of randomness. However, if $p_i(t)$ never cross, this implies that conditionally on G default times are ordered and such order does not depend on G. This means that at most first I_K assets, as given by ordering (5.1), will contribute to the loss on the equity tranche for all Monte Carlo scenarios; (5.2) says exactly that.

In general, when $p_i(t)$ do cross, the last argument does not work: the ordering of default times actually depends on G. What makes (5.1) work are the facts that default times are monotone functions of G, and that for pricing a CDO tranche we are only concerned about the default indicators $\mathbf{1}_{\{\tau\leq t|G\}}$, and not the default times themselves. Since default times do not explicitly appear on in the payoff, they do not matter, only their ordering does. But if we were explicitly exposed to default times, through a more complex payoff or modelling assumptions, (e.g. stochastic recoveries, not independent of the default times), the proof of the proposition would not work.

6 Sensitivity calculation

At first sight, it looks like that (5.2) being linear in all marginal $p_i, i < I_K$, implies that

$$\frac{\partial \mathbb{E}(K-L_t)^+}{\widetilde{p_i}} = \begin{cases} l_i, & \text{if } i < I_K, \\ K - L^{I_K - 1}, & \text{if } i = I_K, \\ 0, & \text{if } i > I_K. \end{cases} \tag{6.1}$$

Because of this, it is seemingly possible to recompute the $\mathbb{E}(K-L_t)^+$ when one of the survival probabilities p_i is substituted by $p_i + \delta p_i$, for arbitrarily large δp_i:

$$\mathbb{E}(K-L_t)^+_{p_i \to p_i + \Delta p_i} = \mathbb{E}(K-L_t)^+_{p_i} + \frac{\partial \mathbb{E}(K-L_t)^+}{\widetilde{p_i}} \Delta p_i. \tag{6.2}$$

This does not work in general, only if the change in the $\widetilde{p_i}$ does not alter ordering of the assets that we introduced in the formulation of **Theorem 5.1**. The original ordering can be violated for the two main reasons: large Δp_i or several names having same initial $\widetilde{p_i}$

Therefore, we do not recommend using the above extensions for the comonotone case, but recompute $EL_t(K)$ directly using (5.2). Such calculation will amount to subtracting the original contribution of the name being bumped, establishing the new order given the bump scenario and recomputing $EL_t(K)$ for the scenario.

7 Application to the nearly comonotone case

Analytical tractability of the comonotone case can be used to stabilize calculation of $\mathbb{E}(K-L_t)^+$ if the copula becomes comonotone for some value of the copula parameter, provided that $\mathbb{E}(K-L_t)^+$ is continuous function of the parameter. In this case, one can interpolate between the comonotone expected loss and another expected loss, obtained for the copula parameter, for which the model still converges reasonably well in reasonable computation time. We will refer to copula parameter in the latter case as the "upper boundary" or "maximum" value of parameter (motivated by the case of Gaussian copula). Denoting such parameter α, and its upper boundary value as $\overline{\alpha}$, and the value of $\mathbb{E}(K-L_t)^+$ in the comonotone case as $\mathbb{E}_c(K-L_t)^+$, we can formalize this as

$$\mathbb{E}_\alpha(K-L_t)^+ = f\left(\alpha; \mathbb{E}_{\overline{\alpha}}(K-L_t)^+, \mathbb{E}_c(K-L_t)^+\right), \tag{7.1}$$

for some interpolation function $f(t;x,y)$.

The interpolation function should be, in principle, selected such that the following three **consistency conditions** on $\mathbb{E}_\alpha(K-L_t)^+$ are fulfilled:

1. monotonicity and convexity in K :
$$\frac{\partial}{\partial K}\mathbb{E}_\alpha(K-L_t)^+ \geq 0,\ \frac{\partial}{\partial K^2}\mathbb{E}_\alpha(K-L_t)^+ \geq 0 \tag{7.2}$$
2. consistent dependency on copula parameter,
$$\text{in case of Gaussian copula, monotonicity in } \rho : \frac{\partial}{\partial \rho}\mathbb{E}_\alpha(K-L_t)^+ \leq 0. \tag{7.3}$$
3. monotonicity in p : $\frac{\partial}{\partial p}\mathbb{E}_\alpha(K-L_t)^+ \geq 0$ (7.4)

For practical purposes, satisfying conditions (7.3) and (7.4) are critical. The main application of our method is stable calibration of the model when market is distressed, although the solution exists. In this case, condition (7.3) warrants that the solution will be found.

Fully satisfying condition (7.2) is most difficult. This condition is a restatement of the fact that $\mathbb{E}_\alpha(K-L_t)$ is taken with respect to a non-negative density. The condition therefore warrants that the expected loss curve as the function of detachment is arbitrage free, which is mostly relevant in pricing tranchelets. Of the two inequalities in (7.2), ensuring monotonicity is fundamental, as it makes values of any tranchelets non-negative. However, if only monotonicity is ensured, it may be the case that there is no loss distribution that could support the positive value of the tranchelets obtained with such approximation. Still, if the main application of our approach is calibration of the base correlations for senior base tranches with well separated detachment points (like 12%-22% or 15%-30% or alike), only satisfying $\frac{\partial}{\partial K}\mathbb{E}_\alpha(K-L_t)^+ \geq 0$ may be a reasonable trade-off between the complexity of the interpolation function and practicality.

Condition (7.4) introduces intertemporal dependency between $\mathbb{E}_\alpha(K-L_t)^+$ for different t. Time does not show up explicitly in our approach, only through the cumulative default probabilities. However, since cumulative default probabilities are non-decreasing in time, it is sufficient to formulate the intertemporal condition only in terms of default probabilities. The nature of this condition is thus equivalent to the $\frac{\partial}{\partial K}\mathbb{E}_\alpha(K-L_t)^+ \geq 0$ condition in term of detachment.

If the copula family with parameter α implies that $\mathbb{E}_\alpha(K-L_t)^+$ is monotone in α,the most general brute force approach would be to interpolate linearly in α:

$$\mathbb{E}_\alpha(K-L_t)^+ = \mathbb{E}_{\alpha_{\max}}(K-L_t)^+ + \frac{\alpha-\alpha_{\max}}{\alpha_{comon}-\alpha_{\max}}\left[\mathbb{E}_{comon}(K-L_t)^+ - \mathbb{E}_{\alpha_{\max}}(K-L_t)^+\right].$$

For particular families it may be possible to derive better approximations. Observe that linear interpolation trivially fulfills conditions (7.2)-(7.4), therefore it may, in principle, serve as a fallback option. As we show below, already for Gaussian copula linear interpolation is not good enough, therefore we will have to deal with a more detailed analysis of behavior of our approximation functions so as to fulfil (7.2) - (7.4).

To compute CDS spread or default sensitivity using (7.1) one will have to first compute unconditional $\mathbb{E}_{\overline{\alpha}}(K-L_t)^+$ in the bumped case and only then apply (7.1). For complex weight functions (e.g. if they explicitly depend on other deal or model characteristics then just the copula parameter), the weight function will have to be selected so as to ensure at least the correct sign of the sensitivities, produced by the approximation.

7.1 One-factor Gaussian copula

As the base correlation framework for Gaussian copula is market standard, at least as the lower level computational engine, deriving the comonotone approximation for Gaussian copula is of highest practical interest. It can be proven that base tranche expected loss monotonically falls when correlation increases (see [BGL08] for references to the proof using the supermodular ordering formalism). Thus, it should be possible to come up with approximation for Gaussian copula for high correlations.

7.1.1 Approximation

Perhaps the simplest approach to come up with an approximation is to consider the LHP case and postulate the corresponding approximation on ρ for the general case. Obviously, this will be the zeroth order approximation in terms of the portfolio diversity (say, characterized by the standard deviation of spreads). Therefore one can argue that whatever more complex approximation it would be possible to come up with, it will contain the LHP as the zeroth order approximation in terms of diversity. Studying higher order corrections are beyond the scope of this work.

Recall that LHP implies that

$$H^2_\rho(K) = \Phi^2\left(-\Phi^{-1}(p), \Phi^{-1}(K), -\sqrt{1-\rho}\right), \tag{7.5}$$
$$H^2_{\rho=1}(K) = \Phi(-\Phi^{-1}(p))\Phi(\Phi^{-1}(K)) = (1-p)K.$$

The expansion of $\Phi^2(x,y,r)$ in r is given by ([AS72], formula (26.3.29), rewritten using our notation):

$$\Phi^2(x,y,r) = \Phi(x)\Phi(y) + \sum_{n=1}^{\infty} \frac{\Phi^{(n)}(-x)\Phi^{(n)}(-y)}{n!} r^n$$

Formula (7.5) implies that $\rho \to 1$ is equivalent to $r \to -1$ in the above. Since

$$\Phi(f(x))' = \frac{1}{\sqrt{2\pi}} f(x)' e^{-\frac{-f(x)^2}{2}} \overset{f(x)=x}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{-x^2}{2}}, \text{ and} \tag{7.6}$$
$$\Phi(f(x))'' = \frac{1}{\sqrt{2\pi}} e^{-\frac{-f(x)^2}{2}} \left(f(x)'' - \left[f(x)'\right]^2\right) \overset{f(x)=x}{=} -x\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \tag{7.7}$$

we obtain for the first three terms:

$$H^2_{\rho\to 1}(K) = (1-p)K - \frac{1}{2\pi} e^{-\frac{\Phi^{-1}(p)^2+\Phi^{-1}(K)^2}{2}} \left(\sqrt{1-\rho} + \frac{1}{2}(1-\rho)\Phi^{-1}(p)\Phi^{-1}(K)\right) \Rightarrow$$
$$\mathbb{E}(K-L_t)^+ \overset{\rho\to 1}{\to} pK + A(K,p)\left(\sqrt{1-\rho} + \frac{1}{2}(1-\rho)\Phi^{-1}(p)\Phi^{-1}(K)\right), \tag{7.8}$$

where we have put

$$A(K,p) = \frac{1}{2\pi} e^{-\frac{\Phi^{-1}(p)^2+\Phi^{-1}(K)^2}{2}},$$

as it does not depend on ρ. Therefore

$$\Delta EL(K,\rho) = \mathbb{E}(K-L_t)^+_\rho - \mathbb{E}(K-L_t)^+_{\rho=1} \tilde{\ } C(K,p)\left(\sqrt{1-\rho} + \frac{1}{2}(1-\rho)\Phi^{-1}(p)\Phi^{-1}(K)\right).$$

This can be postulated as the approximation formula for the general case. The parameter $C(K,p)$ is determined from the boundary condition

$$\Delta EL(K,\rho_{\max}) = \mathbb{E}(K-L_t)^+_{\rho_{\max}} - \mathbb{E}(K-L_t)^+_{\rho=1},$$

where $\mathbb{E}(K-L_t)^+_{\rho=1}$ is to be determined using (5.2) and $\mathbb{E}(K-L_t)^+_{\rho_{\max}}$ is to be evaluated using a standard factor copula technique for baskets, e.g. [ASB03].Thus, if our upper boundary correlation is $\rho_{\max}$, we have

$$\mathbb{E}_\rho(K-L_t)^+ = \mathbb{E}(K-L_t)^+_{\rho=1} + \frac{\mathbb{E}(K-L_t)^+_{\rho_{\max}} - \mathbb{E}(K-L_t)^+_{\rho=1}}{\sqrt{1-\rho_{\max}} + \frac{1}{2}(1-\rho_{\max})\Phi^{-1}(p)\Phi^{-1}(K)}\left(\sqrt{1-\rho} + \frac{1}{2}(1-\rho)\Phi^{-1}(p)\Phi^{-1}(K)\right). \tag{7.9}$$

The reason we consider the first two terms is that the second term can be sufficiently large compared to the first term. The tables low show the ratio of the second to the first term in (7.9), which equals $q(p,K) = \frac{1}{2}\sqrt{1-\rho}\Phi^{-1}(p)\Phi^{-1}(K)$, for the fixed $\rho = \{0.95, 0.90\}$ as a function of p and K, for $p \in [0.0001, 0.2000]$ and $K \in [0, 45]$. The ratio is 0 for $K = 0.5$ and $q(p, 0.5+x) = -q(p, 0.5-x)$, since $\Phi^{-1}(K)$ is odd, so we do not provide values for $K > 0.5$, as they are merely the negated ones below, symmetric about $K = 0.5$.

Second Correction Term as % of First Correction Term

$\rho = 0.95$

	Probability							
K	0.0001	0.0005	0.0010	0.0050	0.0100	0.0500	0.1000	0.2000
5%	68.39%	60.51%	56.83%	47.37%	42.78%	30.25%	23.57%	15.48%
10%	53.29%	47.15%	44.28%	36.91%	33.33%	23.57%	18.36%	12.06%
15%	43.09%	38.13%	35.81%	29.85%	26.96%	19.06%	14.85%	9.75%
20%	34.99%	30.96%	29.08%	24.24%	21.89%	15.48%	12.06%	7.92%
25%	28.05%	24.81%	23.30%	19.42%	17.54%	12.40%	9.66%	6.35%
30%	21.80%	19.29%	18.12%	15.10%	13.64%	9.64%	7.51%	4.93%
35%	16.02%	14.18%	13.31%	11.10%	10.02%	7.09%	5.52%	3.63%
40%	10.53%	9.32%	8.75%	7.30%	6.59%	4.66%	3.63%	2.38%
45%	5.22%	4.62%	4.34%	3.62%	3.27%	2.31%	1.80%	1.18%

$\rho = 0.90$

	Probability							
K	0.0001	0.0005	0.0010	0.0050	0.0100	0.0500	0.1000	0.2000
5%	96.72%	85.58%	80.37%	66.99%	60.50%	42.78%	33.33%	21.89%
10%	75.36%	66.68%	62.62%	52.19%	47.14%	33.33%	25.97%	17.05%
15%	60.95%	53.92%	50.64%	42.21%	38.12%	26.95%	21.00%	13.79%
20%	49.49%	43.79%	41.12%	34.28%	30.96%	21.89%	17.05%	11.20%
25%	39.66%	35.09%	32.96%	27.47%	24.81%	17.54%	13.67%	8.98%
30%	30.84%	27.28%	25.62%	21.36%	19.29%	13.64%	10.63%	6.98%
35%	22.66%	20.05%	18.83%	15.69%	14.17%	10.02%	7.81%	5.13%
40%	14.90%	13.18%	12.38%	10.32%	9.32%	6.59%	5.13%	3.37%
45%	7.39%	6.54%	6.14%	5.12%	4.62%	3.27%	2.55%	1.67%

These numbers mean that for small default probabilities and both small and large detachments, the contribution of the second term is quite significant, about 20% on average for $\rho = 0.9$, and about 30% on average for $\rho = 0.95$.

7.1.2 Satisfying the consistency conditions

We have to make sure that conditions (7.2)-(7.4) are fulfilled. As the last tables hints, approximation (7.9) is not always monotone in all three arguments (K, p and ρ), because the second term changes sign. It would be monotone, if we ignored the second term, making the approximation function linear in $\sqrt{1-\rho}$, hence the whole approximation scheme becoming linear interpolation in terms of $\sqrt{1-\rho}$.

Consider monotonicity in ρ first. approximation (7.9) is monotonic in ρ if

$$sgn\left(\Phi^{-1}(p)\Phi^{-1}(K)\right) \geq 1.$$

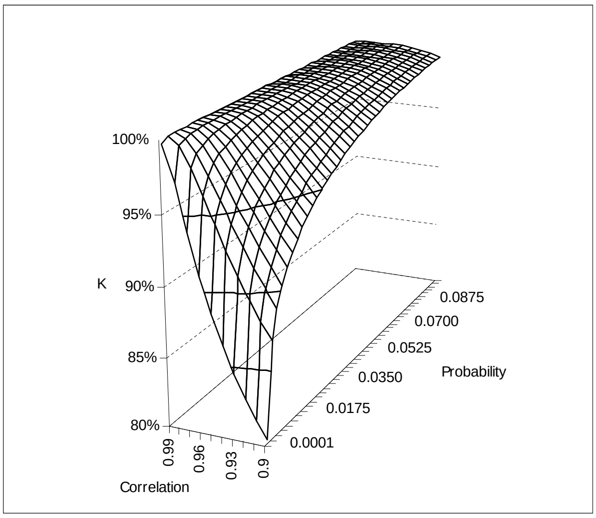
In practical applications, we will almost surely have $p < 0.5$, implying $\Phi^{-1}(p) < 0$. However, since $K \in [0,1]$, we may have to deal with the situation when (7.9) fails to be monotone for senior tranches, when $K > 0.5$ (which, for example, would correspond to tranches on CDX with detachments above 30%, assuming a uniform 40% recovery rate).

For the given p, the critical K (for which (7.9) will fail to be monotone) is given by

$$K^* = \Phi\left(-\frac{1}{\Phi^{-1}(p)\sqrt{1-\rho}}\right). \tag{7.10}$$

Fix $p < 0.5$, then $\Phi^{-1}(p)$ is negative. Also, as we are interested in the cases when $\rho \to 1$, $-1/\left(\Phi^{-1}(p)\sqrt{1-\rho}\right)$ will be a large positive number, implying that K should be close to 100% of the notional. The lower K^* will be realized for smaller p and smaller ρ. The chart below shows $K^*(p,\rho)$, for the most relevant ranges $p \in [0.0001, 0.2000]$ and $\rho \in [0.90, 0.99]$; as we can see $K^* > 0.8$.

Critical Detachment $K^*(p,\rho)$



One way to deal with potential non-monotonicity in ρ is to completely ignore the second term in (7.9) for K's where non-monotonicity can show up in practice. Selection of the cutoff K is can done using (7.10); the simplest rule would be to ignore the second term for $K > 0.5$. As our tests show (see section on Numerical examples), this does not decrease the approximative power of our method much simply because LHP works much better for the senior tranches than for the junior tranches.

Since the second term in (7.9) depends on p and K in exactly the same way, we could also chose to ignore the second order term for $p > 0.5$. However, since such situation is unrealistic, we will always assume that $p < 0.5$ in the sequel.

Thus, our final check is for condition (7.2). For the practical reasons explained above, here we only ensure that (7.9) preserves monotonicity in K, or, equivalently, p, with other parameters fixed. As discussed above, we also assume that $p < 0.5$ and $K < 0.5$.

Observe that (7.9) can be rewritten as

$$F(K) = \mathbb{E}(K - L_t)^+_{\rho=1}\left(1 - \frac{\sqrt{1-\rho} + \frac{1}{2}(1-\rho)\,\Phi^{-1}(p)\Phi^{-1}(K)}{\sqrt{1-\rho_{\max}} + \frac{1}{2}(1-\rho_{\max})\,\Phi^{-1}(p)\Phi^{-1}(K)}\right) + \frac{\sqrt{1-\rho} + \frac{1}{2}(1-\rho)\,\Phi^{-1}(p)\Phi^{-1}(K)}{\sqrt{1-\rho_{\max}} + \frac{1}{2}(1-\rho_{\max})\,\Phi^{-1}(p)\Phi^{-1}(K)}\mathbb{E}(K - L_t)^+_{\rho_{\max}}$$

Proposition 7.1 *Let*

1. $f(x) \geq g(x) \geq 0$, with both functions monotonically increasing and convex

2. $w(x) \in [0, 1]$ and monotonically increasing

Then $h(x) = f(x)w(x) + g(x)(1 - w(x))$ is increasing too.

If in addition to that $f(x)' \geq g(x)' \geq 0$ and $w(x)$ is convex then $f(x)w(x) + g(x)(1 - w(x))$ is also convex.

Proof. Differentiation and application of (1) yields

$$h(x)' = f(x)'w(x) + g(x)'(1 - w(x)) + [f(x) - g(x)]\,w(x)' \geq 0. \tag{7.11}$$

Second differentiation and application of both (1) and (2) yields.

$$h(x)'' = w(x)''\,(f(x) - g(x)) + 2w(x)'\left(f(x)' - g(x)'\right) + w(x)\left(f(x)'' - g(x)''\right) + g(x)'' \geq 0.$$

■

Our goal is to apply the first half of the **Proposition 7.1** to $F(K)$. In the notation of the lemma we have:

$$f(K) = \mathbb{E}(K - L_t)^+_{\rho_{\max}}, \tag{7.12}$$

$$g(K) = \mathbb{E}(K - L_t)^+_{\rho=1}, \tag{7.13}$$

$$w(K) = \frac{\sqrt{1-\rho} + \frac{1}{2}(1-\rho)\,\Phi^{-1}(p)\Phi^{-1}(K)}{\sqrt{1-\rho_{\max}} + \frac{1}{2}(1-\rho_{\max})\,\Phi^{-1}(p)\Phi^{-1}(K)}. \tag{7.14}$$

To apply **Proposition 7.1** we only need to check the behavior of the weight function $w(K)$, as $\mathbb{E}(K - L_t)^+_{\rho_{\max}}$ and $\mathbb{E}(K - L_t)^+_{\rho=1}$ and monotonically increasing and convex by construction.

Proposition 7.2 *If $p < 0.5$ and $K < 0.5$ then $\partial_K w \geq 0$ and $\partial_p w \geq 0$ for each $\rho \in [\rho_{\max}, 1]$.*

Proof. Differentiation by K yields:

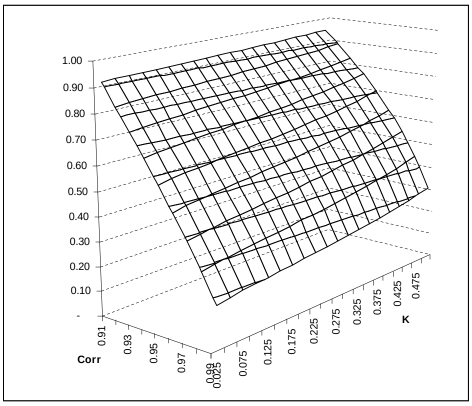
$$\frac{\partial w}{\partial K}=\frac{1}{2}\Phi^{-1}(p)\frac{\left\{\begin{array}{c}(1-\rho)\,\Phi^{-1}(K)'\left(\sqrt{1-\rho_{\max}}+\frac{1}{2}(1-\rho_{\max})\,\Phi^{-1}(p)\Phi^{-1}(K)\right)-\\-(1-\rho_{\max})\,\Phi^{-1}(K)'\left(\sqrt{1-\rho}+\frac{1}{2}(1-\rho)\,\Phi^{-1}(p)\Phi^{-1}(K)\right)\end{array}\right\}}{\left[\sqrt{1-\rho_{\max}}+\frac{1}{2}(1-\rho_{\max})\,\Phi^{-1}(p)\Phi^{-1}(K)\right]^2}$$
$$=\frac{1}{2}\Phi^{-1}(p)\Phi^{-1}(K)'\sqrt{(1-\rho_{\max})(1-\rho)}\frac{\sqrt{1-\rho}-\sqrt{1-\rho_{\max}}}{\left[\sqrt{1-\rho_{\max}}+\frac{1}{2}(1-\rho_{\max})\,\Phi^{-1}(p)\Phi^{-1}(K)\right]^2}. \tag{7.15}$$

Since $p<0.5$, then $\Phi^{-1}(p)<0$; also, clearly $\Phi^{-1}(K)'>0$. Finally, for $\rho\in[\rho_{\max},1]$, $\sqrt{1-\rho}-\sqrt{1-\rho_{\max}}\leq 0$, therefore the above expression is non-negative.

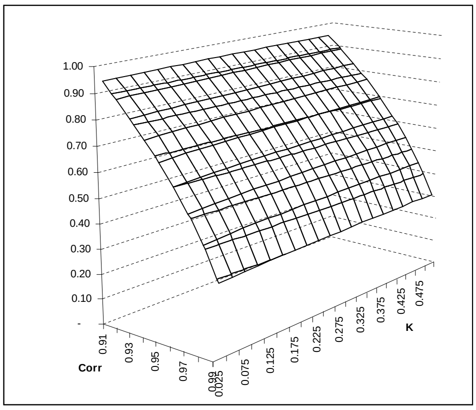
Since w depends on K and p in exactly the same way, the above calculation also implies that $\partial_p w\geq 0$ under the assumption of the proposition. ■

The charts below plot $w(K)$ for $\rho_{\max}=0.9$ and two different and rather extreme values of p.

$p=0.001$



$p=0.4$



Thus, to control monotonicity where it matters we propose the following interpolation form for the expected loss:

$$\mathbb{E}_\rho(K-L_t)^+=\mathbb{E}(K-L_t)^+_{\rho=1}+\frac{\mathbb{E}(K-L_t)^+_{\rho_{\max}}-\mathbb{E}(K-L_t)^+_{\rho=1}}{\sqrt{1-\rho_{\max}}+\theta(0.5-K)(1-\rho_{\max})\,\Phi^{-1}(p)\Phi^{-1}(K)/2}\times$$
$$\left(\sqrt{1-\rho}+\theta(0.5-K)(1-\rho)\,\Phi^{-1}(p)\Phi^{-1}(K)/2\right), \tag{7.16}$$

where $\theta()$ is Heaviside function.

Thus our final approximation (7.16) fulfils conditions (7.3), (7.4) and the monotonicity inequality of (7.2).

7.1.3 Sensitivity calculation

If we ignored the second term in (7.9), then the weight function (7.14) would depend only on ρ. In this case, calculation of sensitivities using (7.16) would amount to calculating sensitivities of $\mathbb{E}(K-L_t)^+_{\rho=1}$ and $\mathbb{E}(K-L_t)^+_{\rho_{\max}}$ and mixing them with the weight. However, as we chose to improve the accuracy of the approximation by introducing the second term, this made the weight function explicitly dependent on p and K. Therefore, sensitivity calculation using (7.16) will require using (7.11) and (7.15) with notation (7.12)- (7.14), to account for the change in the weight function (7.14) due to change in the average portfolio default probability p. As follows from (7.11), the sign of the sensitivity will be preserved by our approximation.

7.1.4 Numerical Examples

In this section we test performance of (7.16) for the combinations of inputs that cover realistic ranges. All tests will be run for 4 scenarios of p_i, such that the average default probabilities $\langle p \rangle^k = \{0.0010, 0.0100, 0.1000, 0.2000\}, k = 1 \dots 4$. It is easy to see, that these values represent cumulative portfolio default probabilities for a portfolio of approximately 100pb credit spread and 40% recovery rate for several intermediary points for calculation $\mathbb{E}(K - L_t)^+_\rho$ for deals with realistic maturities.

On the correlation side, we will consider $\rho_{\max} = \{0.90, 0.95\}$. Other characteristics of the tests are as follows.

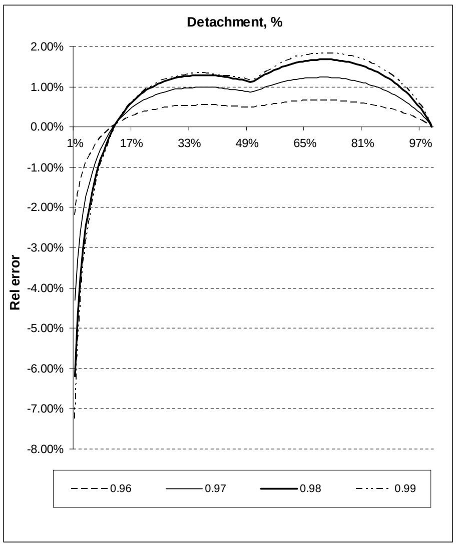
1. Homogenous notional/recovery portfolio of 125 assets.
2. For the test with $p^1 = 0.0010$, the portfolio will be assumed to have uniform spread. For the other three cases of p^k, the default probability of the j-th assets $p^k_j, j = 1 \dots 125$, will be set by the formula
$$p^k_j = p^k + 0.01p^k\,(j - 63)$$
so as to simulate a heterogeneous spread portfolio with default probability linearly dependent on the asset number, such that the average probability is p^k.
3. $\mathbb{E}(K - L_t)^+_{\rho_{\max}}$ is computed using (2.6). The conditional loss distribution is constructed by the recursion method for a uniform loss amount portfolio with heterogenous $p_i(t)$.
4. Copula factor quadrature is uniform on the segment $[-10, 10]$. For computation of $\mathbb{E}(K - L_t)^+_{\rho_{\max}}$ we will use 300 equidistant points in the quadrature, in all calculations. Such simplifying assumptions are made not to address convergence of the quadrature itself for sufficiently high ρ, which is outside the scope of this work.

The quality of approximation (7.16) for $\rho \in (\rho_{\max}, 1)$ will be assessed by comparing its output with with the brute force calculation of $\mathbb{E}(K - L_t)^+_\rho$ with the equidistant 600 point quadrature for the factor on the segment $[-10, 10]$. Such brute force value is denoted below as $\mathbb{E}^{bf}(K - L_t)^+_\rho$.

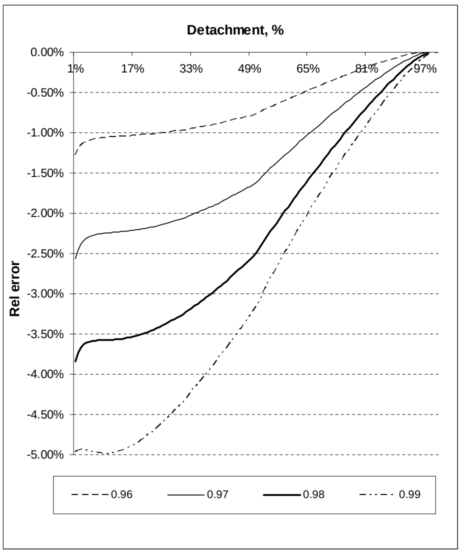
The test results presented below are relative errors of (7.16) for different levels of $K \in [1\%, 100\%]$,
$$\Delta(K; p, \rho_{\max}) = \frac{\mathbb{E}^{bf}(K - L_t)^+_\rho. - \mathbb{E}(K - L_t)^+_\rho}{\mathbb{E}^{bf}(K - L_t)^+_\rho}.$$
To produce the output, K's were discretized with a 1% step.

Below are the charts of $\Delta(K; p, \rho_{\max})$ for $\rho_{\max} = 0.95$.

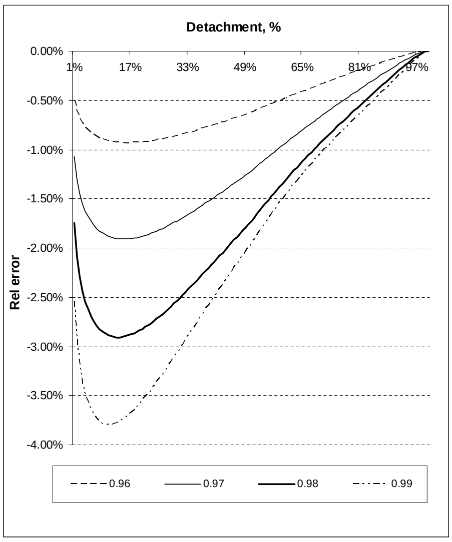
$p = .0010$



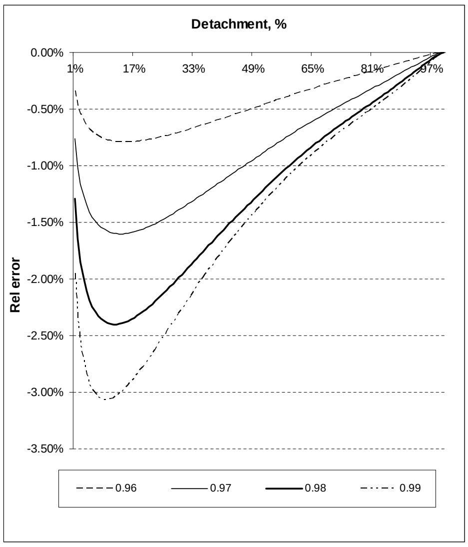
$p = .0100$



$p = 0.1000$



$p = 0.2000$

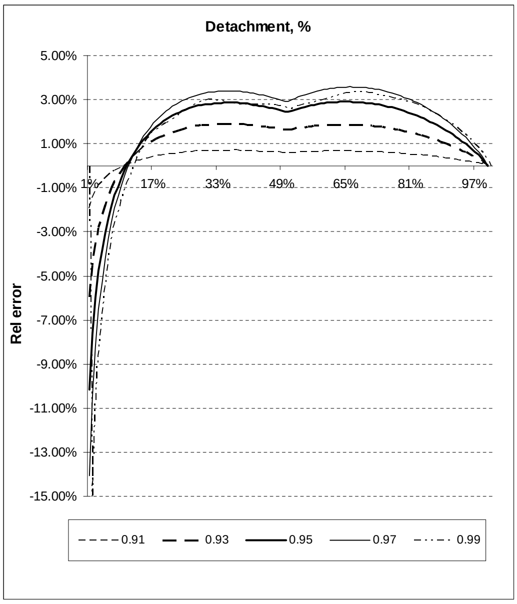


As one can see, approximation (7.16) generally works within 5% relative error. The only exceptions in our test set are the cases for $p = 0.0010$, where $K = 1\%, 2\%, \rho = 99\%$, and $K = 1\%, \rho = 98\%$, where $|\Delta(K; p, \rho_{\max})| \in [5\%, 8\%]$. Clearly, this kind of discrepancy for this kind of detachments and correlations are irrelevant for practical applications.

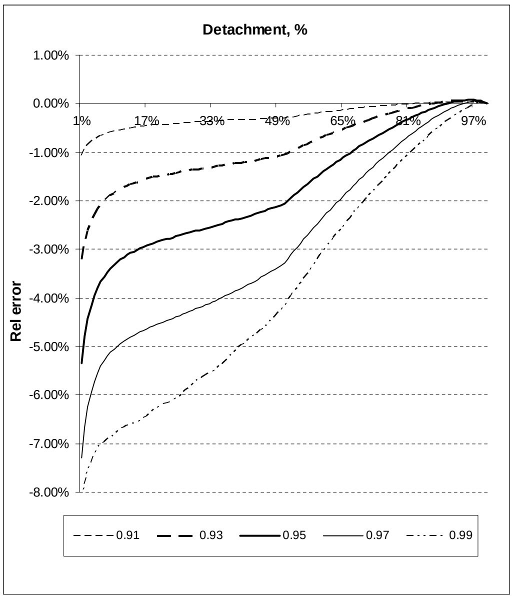
The kink in the graphs around $K = 50\%$ is due to the regime change in (7.16), where we start to ignore the second order (linear) term.

Below are the charts of $\Delta(K; p, \rho_{\max})$ for $\rho_{\max} = 0.90$.

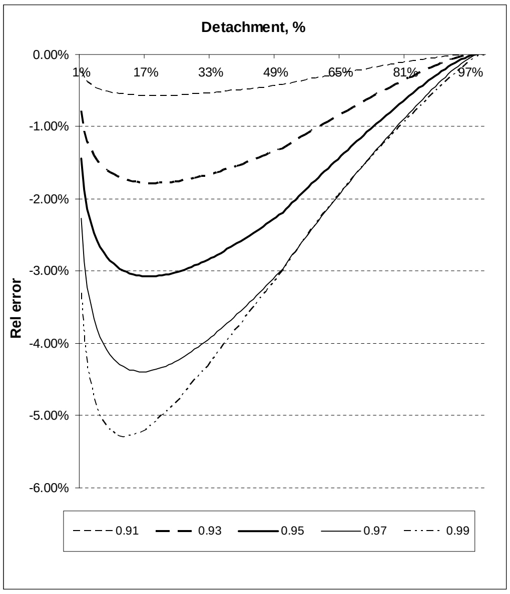
$p = .0010$



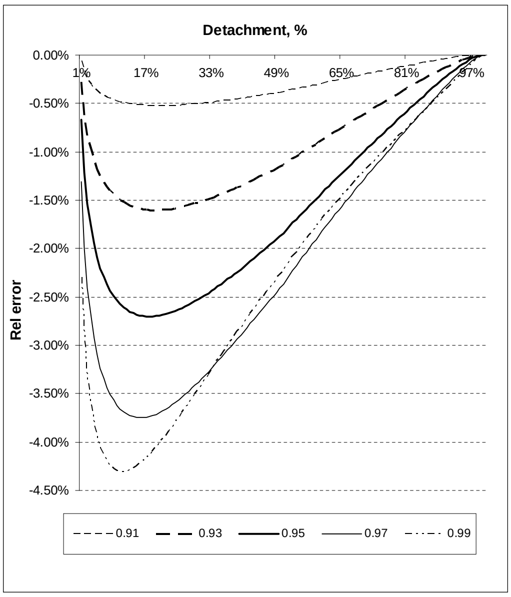
$p = .0100$



$p = 0.1000$



$p = 0.2000$



One can see that the general shapes of the curves is same, although the relative errors are roughly 2x larger than for the case of $\rho_{\max} = 0.95$.

8 Conclusion

Numerical examples in the previous section showed that a two-term approximation reproduces the true value with no more than approximately 5% relative error, uniformly across detachments and average default probabilities for correlations between 95% and 100%. We suppose this is rather good, given that the expected loss function is not differentiable for $\rho = 100\%$, although it is continuos in this point.

The main conclusion from the tests is that $\rho_{\max} = 0.95$ is perhaps the optimal level for the upper correlation to be used in practice.

A necessary extension of the approach will be accounting for the CDS spread inhomogeniety in the portfolio, as this appears to be a major characteristic of the current market situation.

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